PUTNAM TRAINING
NUMBER THEORY
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Remark. This is a list of exercises on Number Theory. —Miguel A. Lerma

Exercises

1. Show that the sum of two consecutive primes is never twice a prime.

2. Can the sum of the digits of a square be (a) 3, (b) 1977?

3. Prove that there are infinitely many prime numbers of the form $4n + 3$.

4. Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every possible integer $n$.

5. Let $p(x)$ be a non-constant polynomial such that $p(n)$ is an integer for every positive integer $n$. Prove that $p(n)$ is composite for infinitely many positive integers $n$. (This proves that there is no polynomial yielding only prime numbers.)

6. Show that if $a^2 + b^2 = c^2$, then $3 | ab$.

7. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ can never be an integer for $n \geq 2$.

8. Let $f(n)$ denote the sum of the digits of $n$. Let $N = 4444^{4444}$. Find $f(f(f(N)))$.

9. Show that there exist 1999 consecutive numbers, each of which is divisible by the cube of some integer greater than 1.

10. Find all triples of positive integers $(a, b, c)$ such that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 2.$$ 

11. Find all positive integer solutions to $abc - 2 = a + b + c$.

12. (USAMO, 1979) Find all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599.$$ 

13. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that for some $k > 0$, $F_k$ is a multiple of $10^{10^{10^{39}}}$.
14. Do there exist 2 irrational numbers $a$ and $b$ greater than 1 such that $\left\lfloor a^m \right\rfloor \neq \left\lfloor b^n \right\rfloor$ for every positive integers $m, n$?

15. The numbers $2^{2005}$ and $5^{2005}$ are written one after the other (in decimal notation). How many digits are written altogether?

16. If $p$ and $p^2 + 2$, are primes show that $p^3 + 2$ is prime.

17. Suppose $n > 1$ is an integer. Show that $n^4 + 4^n$ is not prime.

18. Let $m$ and $n$ be positive integers such that $m < \left\lfloor \sqrt{n} + \frac{1}{2} \right\rfloor$. Prove that $m + \frac{1}{2} < \sqrt{n}$.

19. Prove that the function $f(n) = \left\lfloor n + \sqrt{n} + 1/2 \right\rfloor$ $(n = 1, 2, 3, \ldots)$ misses exactly the squares.

20. Prove that there are no primes in the following infinite sequence of numbers:
   
   \[ \text{1001, 1001001, 1001001001, 1001001001001, \ldots} \]

21. (Putnam 1975, A1.) For positive integers $n$ define $d(n) = n - m^2$, where $m$ is the greatest integer with $m^2 \leq n$. Given a positive integer $b_0$, define a sequence $b_i$ by taking $b_{k+1} = b_k + d(b_k)$. For what $b_0$ do we have $b_i$ constant for sufficiently large $i$?

22. Let $a_n = 10 + n^2$ for $n \geq 1$. For each $n$, let $d_n$ denote the gcd of $a_n$ and $a_{n+1}$. Find the maximum value of $d_n$ as $n$ ranges through the positive integers.

23. Suppose that the positive integers $x, y$ satisfy $2x^2 + x = 3y^2 + y$. Show that $x - y$, $2x + 2y + 1$, $3x + 3y + 1$ are all perfect squares.

24. If $2n + 1$ and $3n + 1$ are both perfect squares, prove that $n$ is divisible by 40.

25. How many zeros does $1000!$ ends with?

26. For how many $k$ is the binomial coefficient $\binom{100}{k}$ odd?

27. Let $n$ be a positive integer. Suppose that $2^n$ and $5^n$ begin with the same digit. What is the digit?

28. Prove that there are no four consecutive non-zero binomial coefficients $\binom{n}{r}$, $\binom{n}{r+1}$, $\binom{n}{r+2}$, $\binom{n}{r+3}$ in arithmetic progression.

29. (Putnam 1995, A1) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another.

30. (Putnam 2003, A1) Let $n$ be a fixed positive integer. How many ways are there to write $n$ as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with $k$ an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: $4, 2+2, 1+1+2, 1+1+1+1$. 
31. (Putnam 2001, B-1) Let $n$ be an even positive integer. Write the numbers 1, 2, \ldots, $n^2$ in the squares of an $n \times n$ grid so that the $k$th row, from right to left is $(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n$.

Color the squares of the grid so that half the squares in each row and in each column are read and the other half are black (a chalkboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers in the black squares.

32. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1s and 0s, beginning and ending with 1?

33. Prove that if $n$ is an integer greater than 1, then $n$ does not divide $2^n - 1$.

34. The digital root of a number is the (single digit) value obtained by repeatedly adding the (base 10) digits of the number, then the digits of the sum, and so on until obtaining a single digit—e.g. the digital root of 65,536 is 7, because $6 + 5 + 5 + 3 + 6 = 25$ and $2 + 5 = 7$. Consider the sequence $a_n = \text{integer part of } 10^n \pi$, i.e.,

$$a_1 = 31, \quad a_2 = 314, \quad a_3 = 3141, \quad a_4 = 31415, \quad a_5 = 314159, \quad \ldots$$

and let $b_n$ be the sequence

$$b_1 = a_1, \quad b_2 = a_1^{a_2}, \quad b_3 = a_1^{a_2^{a_3}}, \quad b_4 = a_1^{a_2^{a_3^{a_4}}}, \quad \ldots$$

Find the digital root of $b_{10^6}$. 
Hints

1. Contradiction.

2. If $s$ is the sum of the digits of a number $n$, then $n - s$ is divisible by 9.

3. Assume that there are finitely many primes of the form $4n + 3$, call $P$ their product, and try to obtain a contradiction similar to the one in Euclid’s proof of the infinitude of primes.

4. Prove that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime.

5. Prove that $p(k)$ divides $p(p(k) + k)$.

6. Study the equation modulo 3.

7. Call the sum $S$ and find the maximum power of 2 dividing each side of the equality

$$n!S = \sum_{k=1}^{n} \frac{n!}{k}.$$

8. $f(n) \equiv n \pmod{9}$.


10. The minimum of $a, b, c$ cannot be very large.

11. Try changing variables $x = a + 1$, $y = b + 1$, $z = c + 1$.

12. Study the equation modulo 16.

13. Use the Pigeonhole Principle to prove that the sequence of pairs $(F_n, F_{n+1})$ is eventually periodic modulo $N = 10^{10^{10}}$.

14. Try $a = \sqrt{6}$, $b = \sqrt{3}$.

15. —

16. If $p$ is an odd number not divisible by 3, then $p^2 \equiv \pm 1 \pmod{6}$.

17. Sophie Germain’s identity: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

18. The number $\sqrt{n}$ is irrational or an integer.

19. If $m \neq \lfloor n + \sqrt{n} + 1/2 \rfloor$, what can we say about $m$?

20. Each of the given numbers can be written $p_n(10^3)$, where $p_n(x) = 1 + x + x^2 + \cdots + x^n$, $n = 1, 2, 3, \ldots$. 
21. Study the cases $b_k = $ perfect square, and $b_k = $ not a perfect square. What can we deduce about $b_{k+1}$ being or not being a perfect square in each case?

22. $\gcd(a, b) = \gcd(a, b-a)$.

23. What is $(x - y)(2x + 2y + 1)$ and $(x - y)(3x + 3y + 1)$?

24. Think modulo 5 and modulo 8.

25. Think of 1000! as a product of prime factors and count the number of 2’s and the number of 5’s in it.

26. Find the exponent of 2 in the prime factorization of $(\binom{n}{k})$.

27. If $N$ begins with digit $a$ then $a \cdot 10^k \leq N < (a+1) \cdot 10^k$.

28. The desired sequence of binomial numbers must have a constant difference.

29. Induction. The base case is $1 = 2^03^0$. The induction step depends on the parity of $n$.
   If $n$ is even, divide by 2. If it is odd, subtract a suitable power of 3.

30. If $0 < k \leq n$, is there any such sum with exactly $k$ terms? How many?

31. Interpret the grid as a 'sum' of two grids, one with the terms of the form $(k-1)n$, and the other one with the terms of the form $1, \ldots, n$.

32. Each of the given numbers can be written $p_n(10^2)$, where $p_n(x) = 1 + x + x^2 + \cdots + x^n$, $n = 1, 2, 3, \ldots$.

33. If $n$ is prime Fermat’s Little Theorem yields the result. Otherwise let $p$ be the smallest prime divisor of $n$.

34. The digital root of a number is its reminder modulo 9. Then show that $a_1^n \ (n = 1, 2, 3, \cdots)$ modulo 9 is periodic.
Solutions

1. If $p$ and $q$ are consecutive primes and $p + q = 2r$, then $r = (p + q)/2$ and $p < r < q$, but there are no primes between $p$ and $q$.

2. (a) No, a square divisible by 3 is also divisible by 9.
(b) Same argument.

3. Assume that the set of primes of the form $4n + 3$ is finite. Let $P$ be their product. Consider the number $N = P^2 - 2$. Note that the square of an odd number is of the form $4n + 1$, hence $P^2$ is of the form $4n + 1$ and $N$ will be of the form $4n + 3$. Now, if all prime factors of $N$ where of the form $4n + 1$, $N$ would be of the form $4n + 1$, so $N$ must have some prime factor $p$ of the form $4n + 3$. So it must be one of the primes in the product $P$, hence $p$ divides $N - P^2 = 2$, which is impossible.

4. That is equivalent to proving that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime for every $n$. These are two possible ways to show it:
- Assume a prime $p$ divides $n^3 + 2n = n(n^2 + 2)$. Then it must divide $n$ or $n^2 + 2$. Writing $n^4 + 3n^2 + 1 = n^2(n^2 + 3) + 1 = (n^2 + 1)(n^2 + 2) - 1$ we see that $p$ cannot divide $n^4 + 3n^2 + 1$ in either case.
- The following identity
  $$(n^2 + 1)(n^4 + 3n^2 + 1) - (n^3 + 2n)^2 = 1$$
  (which can be checked algebraically) shows that any common factor of $n^4 + 3n^2 + 1$ and $n^3 + 2n$ should divide 1, so their gcd is always 1. (Note: if you are wondering how I arrived to that identity, I just used the Euclidean algorithm on the two given polynomials.)

5. Assume $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, with $a_n \neq 0$. We will assume WLOG that $a_n > 0$, so that $p(k) > 0$ for every $k$ large enough—otherwise we can use the argument below with $-p(x)$ instead of $p(x)$.
We have
$$p(p(k) + k) = \sum_{i=0}^{n} a_i [p(k) + k]^i.$$ 
For each term of that sum we have that
$$a_i [p(k) + k]^i = [\text{multiple of } p(k)] + a_i k^i,$$
and the sum of the $a_i k^i$ is precisely $p(k)$, so $p(p(k) + k)$ is a multiple of $p(k)$. It remains only to note that $p(p(k) + k) \neq p(k)$ for infinitely many positive integers $k$, otherwise $p(p(x) + x)$ and $p(x)$ would be the same polynomial, which is easily ruled out for non constant $p(x)$.

6. For any integer $n$ we have that $n^2$ only can be 0 or 1 mod 3. So if 3 does not divide $a$ or $b$ they must be 1 mod 3, and their sum will be 2 modulo 3, which cannot be a square.
7. Assume the sum $S$ is an integer. Let $2^i$ be the maximum power of 2 not larger than $n$, and let $2^j$ be the maximum power of 2 dividing $n!$ Then

$$\frac{n!}{2^j 2^i} S = \sum_{k=1}^{n} \frac{n!}{k 2^{j-i}}.$$ 

For $n \geq 2$ the left hand side is an even number. In the right hand side all the terms of the sum are even integers except the one for $k = 2^i$ which is an odd integer, so the sum must be odd. Hence we have an even number equal to an odd number, which is impossible.

8. Since each digit cannot be greater than 9, we have that $f(n) \leq 9 \cdot (1 + \log_{10} n)$, so in particular $f(N) \leq 9 \cdot (1 + 4444 \cdot \log_{10} 4444) < 9 \cdot (1 + 4444 \cdot 4) = 159993$. Next we have $f(f(N)) \leq 9 \cdot 6 = 54$. Finally among numbers not greater than 54, the one with the greatest sum of the digits is 49, hence $f(f(N))) \leq 4 + 9 = 13$.

Next we use that $n \equiv f(n)$ (mod 9). Since 4444 ≡ 7 (mod 9), then

$$4444^{4444} \equiv 7^{4444} \pmod{9}.$$ 

We notice that the sequence $7^n$ mod 9 for $n = 0, 1, 2, \ldots$ is 1, 7, 4, 1, 7, 4, \ldots, with period 3. Since 4444 ≡ 1 (mod 3), we have $7^{4444} \equiv 7^1$ (mod 9), hence $f(f(N))) \equiv 7$ (mod 9). The only positive integer not greater than 13 that is congruent with 7 modulo 9 is 7, hence $f(f(N))) = 7$.

9. Pick 1999 different prime numbers $p_1, p_2, \ldots, p_{1999}$ (we can do that because the set of prime numbers is infinite) and solve the following system of 1999 congruences:

$$\begin{align*}
x &\equiv 0 \pmod{p_1^3} \\
x &\equiv -1 \pmod{p_2^3} \\
x &\equiv -2 \pmod{p_3^3} \\
&\vdots \\
x &\equiv -1998 \pmod{p_{1999}^3}
\end{align*}$$

According to the Chinese Remainder Theorem, that system of congruences has a solution $x$ (modulo $M = p_1^3 \ldots p_{1999}^3$). For $k = 1, \ldots, 1999$ we have that $x + k \equiv 0 \pmod{p_k^3}$, hence $x + k$ is in fact a multiple of $p_k^3$.

10. Assume $a \geq b \geq c$. Then

$$2 = \left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) \left(1 + \frac{1}{c}\right) \leq \left(1 + \frac{1}{c}\right)^3.$$ 

From here we get that $c < 4$, so its only possible values are $c = 1, 2, 3$.

For $c = 1$ we get $(1 + 1/c) = 2$, hence

$$\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) = 1,$$

which is impossible.
For $c = 2$ we have $(1 + 1/c) = 3/2$, hence
\[
\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) = \frac{4}{3},
\]
and from here we get
\[
a = \frac{3(b + 1)}{b - 3},
\]
with solutions $(a, b) = (15, 4), (9, 5)$ and $(7, 6)$.
Finally for $c = 3$ we have $1 + 1/c = 1 + 1/3 = 4/3$, hence
\[
\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) = \frac{3}{2}.
\]
So
\[
a = \frac{2(b + 1)}{b - 2}.
\]
The solutions are $(a, b) = (8, 3)$ and $(5, 4)$.
So the complete set of solutions verifying $a \geq b \geq c$ are
\[(a, b, c) = (15, 4, 2), (9, 5, 2), (7, 6, 2), (8, 3, 3), (5, 4, 3).
\]
The rest of the triples verifying the given equation can be obtained by permutations of $a, b, c$.

11. The change of variables $x = a + 1$, $y = b + 1$, $z = c + 1$, transforms the equation into the following one:
\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.
\]
Assuming $x \leq y \leq z$ we have that $x \leq 3 \leq z$.
For $x = 1$ the equation becomes
\[
\frac{1}{y} + \frac{1}{z} = 0.
\]
which is impossible.
For $x = 2$ we have
\[
\frac{1}{y} + \frac{1}{z} = \frac{1}{2},
\]
or
\[
z = \frac{2y}{y - 2},
\]
with solutions $(y, z) = (3, 6)$ and $(4, 4)$.
For $x = 3$ the only possibility is $(y, z) = (3, 3)$.
So the list of solutions is
\[(x, y, z) = (2, 3, 6), (2, 4, 4), (3, 3, 3),
\]
and the ones obtained by permuting $x, y, z$.
With the original variables the solutions are (except for permutations of variables);
\[(a, b, c) = (1, 2, 5), (1, 3, 3), (2, 2, 2).
\]
12. We look at the equation modulo 16. First we notice that $n^4 \equiv 0 \text{ or } 1 \pmod{16}$ depending on whether $n$ is even or odd. On the other hand $1599 \equiv 15 \pmod{16}$. So the equation can be satisfied only if the number of odd terms in the LHS is 15 modulo 16, but that is impossible because there are only 14 terms in the LHS. Hence the equation has no solution.

13. Call $N = 10^{10^{10}}$, and consider the sequence $a_n = \text{remainder of dividing } F_n \text{ by } N$. Since there are only $N^2$ pairs of non-negative integers less than $N$, there must be two identical pairs $(a_i, a_{i+1}) = (a_j, a_{j+1})$ for some $0 \leq i < j$. Let $k = j - i$. Since $a_{n+2} = a_{n+1} + a_n$ and $a_{n-1} = a_{n+1} - a_n$, by induction we get that $a_n = a_{n+k}$ for every $n \geq 0$, so in particular $a_k = a_0 = 0$, and this implies that $F_k$ is a multiple of $N$. (In fact since there are $N^2 + 1$ pairs $(a_i, a_{i+1})$, for $i = 0, 1, \ldots, N^2$, we can add the restriction $0 \leq i < j \leq N$ above and get that the result is true for some $k$ such that $0 < k \leq N^2$.)

14. The answer is affirmative. Let $a = \sqrt{6}$ and $b = \sqrt{3}$. Assume $[a^m] = [b^n] = k$ for some positive integers $m, n$. Then, $k^2 \leq 6^m < (k + 1)^2 = k^2 + 2k + 1$, and $k^2 \leq 3^n < (k + 1)^2 = k^2 + 2k + 1$. Hence, subtracting the inequalities and taking into account that $n > m$:

$$2k \geq |6^m - 3^n| = 3^m|2^m - 3^{n-m}| \geq 3^m.$$ 

Hence $\frac{9^m}{4} \leq k^2 \leq 6^m$, which implies $\frac{1}{4} \leq (\frac{3}{2})^m$. This holds only for $m = 1, 2, 3$. This values of $m$ can be ruled out by checking the values of

$$[a] = 2, \quad [a^2] = 6, \quad [a^3] = 14,$$


Hence, $[a^m] \neq [b^n]$ for every positive integers $m, n$.

15. There are integers $k, r$ such that $10^k < 2^{2005} < 10^{k+1}$ and $10^r < 5^{2005} < 10^{r+1}$. Hence $10^{k+r} < 10^{2005} < 10^{k+r+2}$, $k + r + 1 = 2005$. Now the number of digits in $2^{2005}$ is $k + 1$, and the number of digits in $5^{2005}$ is $r + 1$. Hence the total number of digits is $2^{2005}$ and $5^{2005}$ is $k + r + 2 = 2006$.

16. For $p = 2, p^2 + 2 = 6$ is not prime.

For $p = 3, p^2 + 2 = 11$, and $p^3 + 2 = 29$ are all prime and the statement is true.

For prime $p > 3$ we have that $p$ is an odd number not divisible by 3, so it is congruent to $\pm 1$ modulo 6. Hence $p^2 + 2 \equiv 3 \pmod{6}$ is multiple of 3 and cannot be prime.

17. If $n$ is even then $n^4 + 4^n$ is even and greater than 2, so it cannot be prime.

If $n$ is odd, then $n = 2k + 1$ for some integer $k$, hence $n^4 + 4^n = n^4 + 4 \cdot (2k)^4$. Next, use Sophie Germain’s identity: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

18. From the hypothesis we have that $m + 1 \leq \lfloor \sqrt{n} + \frac{1}{2} \rfloor \leq \sqrt{n} + \frac{1}{2}$. But the second inequality must be strict because $\sqrt{n}$ is irrational or an integer, and consequently $\sqrt{n} + \frac{1}{2}$ cannot be an integer. From here the desired result follows.
19. Assume $m \neq \lfloor n + \sqrt{n + 1/2} \rfloor$ for every $n = 1, 2, 3, \ldots$ Then for some $n$, $f(n) < m < f(n + 1)$. The first inequality implies

$$n + \sqrt{n} + \frac{1}{2} < m.$$ 

The second inequality implies $m + 1 < f(n + 1)$, and

$$m + 1 < n + 1 + \sqrt{n + 1} + \frac{1}{2}$$

(Note that equality is impossible because the right hand side cannot be an integer.) Hence

$$\sqrt{n} < m - n - \frac{1}{2} < \sqrt{n + 1},$$

$$n < (m - n)^2 - (m - n) + \frac{1}{4} < n + 1$$

$$n - \frac{1}{4} < (m - n)^2 - (m - n) < n + \frac{3}{4}$$

$$(m - n)^2 - (m - n) = n.$$ 

$$m = (m - n)^2.$$ 

So, $m$ is a square.

We are not done yet, since we still must prove that $f(n)$ misses all the squares. To do so we use a counting argument. Among all positive integers $\leq k^2 + k$ there are exactly $k$ squares, and exactly $k^2$ integers of the form $f(n) = [n + \sqrt{n + 1/2}]$. Hence $f(n)$ is the $n$th non square.

Another way to express it: in the set $A(k) = \{1, 2, 3, \ldots, k^2 + k\}$ consider the two subsets $S(k) =$ squares in $A(k)$, and $N(k) =$ integers of the form $f(n) = [n + \sqrt{n + 1/2}]$ in $A(k)$. The set $S(k)$ has $k$ elements, $N(k)$ has $k^2$ elements, and $A(k) = S(k) \cup N(k)$. Since

$$|S(k) \cup N(k)| = |S(k)| + |N(k)| - |S(k) \cap N(k)|$$

we get that $|S(k) \cap N(k)| = 0$, i.e, $S(k) \cap N(k)$ must be empty.

20. Each of the given numbers can be written

$$1 + 1000 + 1000^2 + \cdots + 1000^n = p_n(10^3)$$

where $p_n(x) = 1 + x + x^2 + \cdots + x^n$, $n = 1, 2, 3, \ldots$. We have $(x - 1)p_n(x) = x^{n+1} - 1$. 

If we set $x = 10^3$, we get:

$$999 \cdot p_n(10^3) = 10^{3(n+1)} - 1 = (10^{n+1} - 1)(10^{2(n+1)} + 10^{n+1} + 1).$$

If $p_n(10^3)$ were prime it should divide one of the factors on the RHS. It cannot divide $10^{n+1} - 1$, because this factor is less than $p_n(10^3)$, so $p_n(10^3)$ must divide the other factor. Hence $10^{n+1} - 1$ must divide 999, but this is impossible for $n > 2$. In only remains to check the cases $n = 1$ and $n = 2$. But $1001 = 7 \cdot 11 \cdot 13$, and $1001001 = 3 \cdot 333667$, so they are not prime either.
21. We will prove that the sequence is eventually constant if and only if \( b_0 \) is a perfect square.

The “if” part is trivial, because if \( b_k \) is a perfect square then \( d(b_k) = 0 \), and \( b_{k+1} = b_k \).

For the “only if” part assume that \( b_k \) is not a perfect square. Then, suppose that \( r^2 < b_k < (r+1)^2 \). Then, \( d(b_k) = b_k - r^2 \) is in the interval \([1, 2r]\), so \( b_{k+1} = r^2 + 2d(b_k) \) is greater than \( r^2 \) but less than \((r+2)^2\), and not equal to \((r+1)^2\) by parity. Thus \( b_{k+1} \) is also not a perfect square, and is greater than \( b_k \). So, if \( b_0 \) is not a perfect square, no \( b_k \) is a perfect square and the sequence diverges to infinity.

22. The answer is 41. In fact, we have:

\[
\gcd(a_n, a_{n+1}) = \gcd(a_n, a_{n+1} - a_n) = \gcd(n^2 + 10, 2n + 1) = \cdots
\]

(since \( 2n + 1 \) is odd we can multiply the other argument by 4 without altering the \( \gcd \))

\[
\cdots = \gcd(4n^2 + 40, 2n + 1) = \gcd((2n + 1)(2n - 1) + 41, 2n + 1) = \gcd(41, 2n + 1) \leq 41.
\]

The maximum value is attained e.g. at \( n = 20 \).

23. The given condition implies:

\[
(x - y)(2x + 2y + 1) = y^2.
\]

Since the right hand side is a square, to prove that the two factors on the left hand side are also squares it suffices to prove that they are relatively prime. In fact, if \( p \) is a prime number dividing \( x - y \) then it divides \( y^2 \) and consequently it divides \( y \). So \( p \) also divides \( x \), and \( x + y \). But then it cannot divide \( 2x + 2y + 1 \).

An analogous reasoning works using the following relation, also implied the given condition:

\[
(x - y)(3x + 3y + 1) = x^2.
\]

24. It suffices to prove that \( n \) is a multiple of 5 and 8, in other words, that \( n \equiv 0 \pmod{5} \), and \( n \equiv 0 \pmod{8} \).

We first think modulo 5. Perfect squares can be congruent to 0, 1, or 4 modulo 5 only. We have:

\[
\begin{align*}
2n + 1 &\equiv 0 \pmod{5} \implies n \equiv 2 \pmod{5} \\
2n + 1 &\equiv 1 \pmod{5} \implies n \equiv 0 \pmod{5} \\
2n + 1 &\equiv 4 \pmod{5} \implies n \equiv 4 \pmod{5} \\
3n + 1 &\equiv 0 \pmod{5} \implies n \equiv 3 \pmod{5} \\
3n + 1 &\equiv 1 \pmod{5} \implies n \equiv 0 \pmod{5} \\
3n + 1 &\equiv 4 \pmod{5} \implies n \equiv 1 \pmod{5}.
\end{align*}
\]

So the only possibility that can make both \( 2n + 1 \) and \( 3n + 1 \) perfect squares is \( n \equiv 0 \pmod{5} \), i.e., \( n \) is a multiple of 5.
Next, we think modulo 8. Perfect squares can only be congruent to 0, 1, or 4 modulo 8, and we have:

\[ 3n + 1 \equiv 0 \pmod{8} \implies n \equiv 5 \pmod{8} \]
\[ 3n + 1 \equiv 1 \pmod{8} \implies n \equiv 0 \pmod{8} \]
\[ 3n + 1 \equiv 4 \pmod{8} \implies n \equiv 1 \pmod{8}. \]

The possibilities \( n \equiv 5 \pmod{8} \) and \( n \equiv 1 \pmod{8} \) can be ruled out because \( n \) must be even. In fact, if \( 2n + 1 = a^2 \), then \( a \) is odd, and \( 2n = a^2 - 1 = (a + 1)(a - 1) \). Since \( a \) is odd we have that \( a - 1 \) and \( a + 1 \) are even, so \( 2n \) must be a multiple of 4, consequently \( n \) is even. So, we have that the only possibility is \( n \equiv 0 \pmod{8} \), i.e., \( n \) is a multiple of 8.

Since \( n \) is a multiple of 5 and 8, it must be indeed a multiple of 40, QED.

25. The prime factorization of 1000! contains more 2’s than 5’s, so the number of zeros at the end of 1000! will equal the exponent of 5. That will be equal to the number of multiples of 5 in the sequence 1, 2, 3, \ldots, 1000, plus the number of multiples of \( 5^2 = 25 \), plus the number of multiples of \( 5^3 = 125 \), and the multiples of \( 5^4 = 625 \), in total:

\[
\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 200 + 40 + 8 + 1 = 249.
\]

So 1000! ends with 249 zeros.

26. The answer is 8.

More generally, for any given positive integer \( n \), the number of binomial coefficients \( \binom{n}{k} \) that are odd equals 2 raised to the number of 1’s in the binary representation of \( n \)—so, for \( n = 100 \), with binary representation 1100100 (three 1’s), the answer is \( 2^3 = 8 \). We prove it by induction in the number \( s \) of 1’s in the binary representation of \( n \).

- Basis step: If \( s = 1 \), then \( n \) is a power of 2, say \( n = 2^r \). Next, we use that the exponent of 2 in the prime factorization of \( m! \) is

\[
\sum_{i \geq 1} \left\lfloor \frac{m}{2^i} \right\rfloor,
\]

where \( \lfloor x \rfloor \) = greatest integer \( \leq x \). Since \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \), the exponent of 2 in the prime factorization of \( \binom{n}{k} \) is

\[
\sum_{i \geq 1} \left\lfloor \frac{2^r}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{2^r - k}{2^i} \right\rfloor = \sum_{i=1}^{r} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i=1}^{r} \left\lfloor \frac{k}{2^i} \right\rfloor,
\]

where \( \lfloor x \rfloor \) = least integer \( \geq x \). The right hand side is the number of values of \( i \) in the interval from 1 to \( r \) for which \( \frac{k}{2^i} \) is not an integer. If \( k = 0 \) or \( k = 2^r \) then the expression is 0, i.e., \( \binom{2^r}{k} \) is odd. Otherwise, for \( 0 < k < 2^r \), the right hand side is strictly positive (at least \( k/2^r \) is not an integer), and in that case \( \binom{2^r}{k} \) is even. So the
number of values of $k$ for which \( \binom{r}{k} \) is odd is $2 = 2^1$. This sets the basis step of the induction process.

- Induction step: Assume the statement is true for a given $s \geq 1$, and assume that the number of values of $k$ for which the number of 1's in the binary representation of $n$ is $s+1$, where $0 < n' < 2^r$ and $n'$ has $s$ 1's in its binary representation. By induction hypothesis the number of values of $k$ for which \( \binom{n}{k} \) is odd is $2^{s+1}$. To do so we will study the parity of \( \binom{n}{k} \) in three intervals, namely $0 \leq k \leq n'$, $n' < k < 2^r$, and $2^r \leq k \leq n$.

1. For every $k$ such that $0 \leq k \leq n'$, \( \binom{n}{k} \) and \( \binom{n'}{k} \) have the same parity. In fact, using again the above formula to determine the exponent of 2 in the prime factorization of binomial coefficients, we get

   \[
   \sum_{i \geq 1} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{n-k}{2^i} \right\rfloor = \sum_{i \geq 1} \left\lfloor \frac{2^r + n'}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{n-k}{2^i} \right\rfloor
   \]

   \[
   = \sum_{i \geq 1} \left( 2^{r-i} + \left\lfloor \frac{n'}{2^i} \right\rfloor \right) - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{n-k}{2^i} \right\rfloor
   \]

   \[
   = \sum_{i \geq 1} \left\lfloor \frac{n'}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{n-k}{2^i} \right\rfloor.
   \]

   Hence, the number of values of $k$ in the interval from 0 to $n'$ for which \( \binom{n}{k} \) is odd is $2^s$.

2. If $n' < k < 2^r$, then \( \binom{n}{k} \) is even. In fact, we have that the power of 2 in the prime factorization of \( \binom{n}{k} \) is:

\[
\sum_{i \geq 1} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{n-k}{2^i} \right\rfloor = \sum_{i = 1}^{r} \left( \left\lfloor \frac{n}{2^i} \right\rfloor - \left\lfloor \frac{k}{2^i} \right\rfloor - \left\lfloor \frac{n-k}{2^i} \right\rfloor \right).
\]

using \( [x] + [y] \leq [x+y] \), and given that $n/2^i = k/2^i + (n-k)/2^i$, we see that all terms of the sum on the right hand side are nonnegative, and all we have to show is that at least one of them is strictly positive. That can be accomplished by taking $i = r$. In fact, we have $2^r < n < 2^{r+1}$, hence $1 < n/2^r < 2$, $[n/2^r] = 1$. Also, $0 < k < 2^r$, hence $0 < k/2^r < 1$, $[k/2^r] = 0$. And $2^r = n - n' > n - k > 0$, so $0 < (n-k)/2^r < 1$, $[(n-k)/2^r] = 0$. Hence,

\[
\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{k}{2^r} \right\rfloor - \left\lfloor \frac{n-k}{2^r} \right\rfloor = 1 - 0 - 0 = 1 > 0.
\]
(3) If $2^r \leq k \leq n$, then letting $k' = n - k$ we have that $0 \leq k' \leq n'$, and $\binom{n}{k} = \binom{n}{k'}$, and by (1), the number of values of $k$ in the interval from $2^r$ to $n$ for which $\binom{n}{k}$ is odd is $2^s$.

The three results (1), (2) and (3) combined show that the number of values of $k$ for which $\binom{n}{k}$ is odd is $2^s = 2^s + 1$. This completes the induction step, and the result is proved.

27. The answer is 3.

Note that $2^5 = 32$, $5^5 = 3125$, so 3 is in fact a solution. We will prove that it is the only solution.

Let $d$ be the common digit at the beginning of $2^n$ and $5^n$. Then

$$d \cdot 10^r \leq 2^n < (d + 1) \cdot 10^r,$$

$$d \cdot 10^s \leq 5^n < (d + 1) \cdot 10^s$$

for some integers $r, s$. Multiplying the inequalities we get

$$d^2 10^{r+s} \leq 10^n \leq (d + 1)^2 10^{r+s},$$

$$d^2 \leq 10^{r+s} < (d + 1)^2,$$

so $d$ is such that between $d^2$ and $(d + 1)^2$ there must be a power of 10. The only possible solutions are $d = 1$ and $d = 3$. The case $d = 1$ can be ruled out because that would imply $n = r + s$, and from the inequalities above would get

$$5^r \leq 2^s < 2 \cdot 5^r,$$

$$2^s \leq 5^r < 2 \cdot 2^s,$$

hence $2^s = 5^r$, which is impossible unless $r = s = 0$ (implying $n = 0$).

Hence, the only possibility is $d = 3$.

28. Assume that the given binomial coefficients are in arithmetic progression. Multiplying each binomial number by $(k + 3)!(n - k)!$ and simplifying we get that the following numbers are also in arithmetic progression:

$$(k + 1)(k + 2)(k + 3),$$

$$(n - k)(k + 2)(k + 3),$$

$$(n - k)(n - k - 1)(k + 3),$$

$$(n - k)(n - k - 1)(n - k - 2).$$

Their differences are

$$(n - 2k - 1)(k + 2)(k + 3),$$

$$(n - k)(n - 2k - 3)(k + 3),$$

$$(n - k)(n - k - 1)(n - 2k - 5).$$

Writing that they must be equal we get a system of two equations:

$$\begin{cases}
    n^2 - 4kn - 5n + 4k^2 + 8k + 2 = 0 \\
    n^2 - 4kn - 9n + 4k^2 + 16k + 14 = 0
\end{cases}$$
Subtracting both equations we get
\[ 4n - 8k - 12 = 0, \]
i.e., \( n = 2k + 3 \), so the four binomial numbers should be of the form
\[
\binom{2k + 3}{k}, \quad \binom{2k + 3}{k + 1}, \quad \binom{2k + 3}{k + 2}, \quad \binom{2k + 3}{k + 3}.
\]
However
\[ \binom{2k + 3}{k} < \binom{2k + 3}{k + 1} = \binom{2k + 3}{k + 2} > \binom{2k + 3}{k + 3}, \]
so they cannot be in arithmetic progression.

- Remark: There are sets of three consecutive binomial numbers in arithmetic progression, e.g.: \( \binom{7}{1} = 7, \binom{7}{2} = 21, \binom{7}{3} = 35 \).

29. We proceed by induction, with base case \( 1 = 2^03^0 \). Suppose all integers less than \( n - 1 \) can be represented. If \( n \) is even, then we can take a representation of \( n = 2k \) and multiply each term by 2 to obtain a representation of \( n \). If \( n \) is odd, put \( m = \lfloor \log_3 n \rfloor \), so that \( 3^m \leq n < 3^{m+1} \). If \( 3^m = n \), we are done. Otherwise, choose a representation \( (n - 3^m)/2 = s_1 + \cdots + s_k \) in the desired form. Then
\[ n = 3^m + 2s_1 + \cdots + 2s_k, \]
and clearly none of the \( 2s_i \) divide each other or \( 3^m \). Moreover, since \( 2s_i \leq n - 3^m < 3^{m+1} - 3^m \), we have \( s_i < 3^m \), so \( 3^m \) cannot divide \( 2s_i \) either. Thus \( n \) has a representation of the desired form in all cases, completing the induction.

30. There are \( n \) such sums. More precisely, there is exactly one such sum with \( k \) terms for each of \( k = 1, \ldots, n \) (and clearly no others). To see this, note that if \( n = a_1 + a_2 + \cdots + a_k \) with \( a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1 \), then
\[ ka_1 = a_1 + a_1 + \cdots + a_1 \leq n \leq a_1 + (a_1 + 1) + \cdots + (a_1 + 1) = ka_1 + k - 1. \]
However, there is a unique integer \( a_1 \) satisfying these inequalities, namely \( a_1 = \lfloor n/k \rfloor \).
Moreover, once \( a_1 \) is fixed, there are \( k \) different possibilities for the sum \( a_1 + a_2 + \cdots + a_k \):
if \( i \) is the last integer such that \( a_i = a_1 \), then the sum equals \( ka_1 + (i - 1) \). The possible values of \( i \) are \( 1, \ldots, k \), and exactly one of these sums comes out equal to \( n \), proving our claim.

31. Let \( R \) (resp. \( B \)) denote the set of red (resp. black) squares in such a coloring, and for \( s \in R \cup B \), let \( f(s) = g(s) + 1 \) denote the number written in square \( s \), where \( 0 \leq f(s), g(s) \leq n - 1 \). Then it is clear that the value of \( f(s) \) depends only on the row of \( s \), while the value of \( g(s) \) depends only on the column of \( s \). Since every row contains exactly \( n/2 \) elements of \( R \) and \( n/2 \) elements of \( B \),
\[ \sum_{s \in R} f(s) = \sum_{s \in B} f(s). \]
Similarly, because every column contains exactly \( n/2 \) elements of \( R \) and \( n/2 \) elements of \( B \),

\[
\sum_{s \in R} g(s) = \sum_{s \in B} g(s).
\]

It follows that

\[
\sum_{s \in R} f(s) + g(s) + 1 = \sum_{s \in B} f(s) + g(s) + 1,
\]

as desired.

32. The answer is only 101.

Each of the given numbers can be written

\[
1 + 100 + 100^2 + \cdots + 100^n = p_n(10^2),
\]

where \( p_n(x) = 1 + x + x^2 + \cdots + x^n \), \( n = 1, 2, 3, \ldots \). We have \((x - 1)p_n(x) = x^{n+1} - 1\).

If we set \( x = 10^2 \), we get

\[
99 \cdot p_n(10^2) = 10^{2(n+1)} - 1 = (10^n - 1)(10^{n+1} + 1).
\]

If \( p_n(10^2) \) is prime it must divide one of the factors of the RHS. It cannot divide \( 10^n - 1 \) because this factor is less than \( p_n(10^2) \), so \( p_n(10^2) \) must divide the other factor. Hence \( 10^{n+1} - 1 \) must divide 99. This is impossible for \( n \geq 2 \). In only remains to check the case \( n = 1 \). In this case we have \( p_1(10^2) = 101 \), which is prime.

33. By contradiction. Assume \( n \) divides \( 2^n - 1 \) (note that this implies that \( n \) is odd). Let \( p \) be the smallest prime divisor of \( n \), and let \( n = p^k \cdot m \), where \( p \) does not divide \( m \). Since \( n \) is odd we have that \( p \neq 2 \). By Fermat’s Little Theorem we have \( 2^{p-1} \equiv 1 \pmod p \). Also by Fermat’s Little Theorem, \( (2^{mp^k-1})^{p-1} \equiv 1 \pmod p \), hence \( 2^n = 2^{mp^k} = (2^{p^k-1}m)^{p-1} \cdot 2^{p^k-1}m \equiv 2^{p^k-1}m \pmod p \). Repeating the argument we get \( 2^n = 2^{mp^k} \equiv 2^{p^k-1}m \equiv 2^{p^k-2}m \equiv \cdots \equiv 2^m \pmod p \). Since by hypothesis \( 2^n \equiv 1 \pmod p \), we have that \( 2^n \equiv 1 \pmod p \).

Next we use that if \( 2^n \equiv 1 \pmod p \), and \( 2^b \equiv 1 \pmod p \), then \( 2^{\gcd(a,b)} \equiv 1 \pmod p \). If \( g = \gcd(n, p - 1) \), then we must have \( 2^g \equiv 1 \pmod p \). But since \( p \) is the smallest prime divisor of \( n \), and all prime divisors of \( p - 1 \) are less than \( p \), we have that \( n \) and \( p \) do not have common prime divisors, so \( g = 1 \), and consequently \( 2^g = 2 \), contradicting \( 2^g \equiv 1 \pmod p \).

34. In spite of its apparent complexity this problem is very easy, because the digital root of \( b_n \) becomes a constant very quickly. First note that the digital root of a number \( a \) is just the remainder \( r \) of \( a \) modulo 9, and the digital root of \( a^n \) will be the remainder of \( r^n \) modulo 9.

For \( a_1 = 31 \) we have

- digital root of \( a_1 = \) digital root of 31 = 4;
- digital root of \( a_1^2 = \) digital root of 4^2 = 7;
- digital root of \( a_1^3 = \) digital root of 4^3 = 1;
- digital root of \( a_1^4 = \) digital root of 4^4 = 4;
and from here on it repeats with period 3, so the digital root of $a_n^r$ is 1, 4, and 7 for remainder modulo 3 of $n$ equal to 0, 1, and 2 respectively.

Next, we have $a_2 = 314 \equiv 2 \pmod{3}$, $a_2^2 \equiv 2^2 \equiv 1 \pmod{3}$, $a_2^3 \equiv 2^3 \equiv 2 \pmod{3}$, and repeating with period 2, so the remainder of $a_2^r$ depends only on the parity of $n$, with $a_2^r \equiv 1 \pmod{3}$ if $n$ is even, and $a_2^r \equiv 2 \pmod{3}$ if $n$ is odd.

And we are done because $a_3$ is odd, and the exponent of $a_2$ in the power tower defining $b_n$ for every $n \geq 3$ is odd, so the remainder modulo 3 of the exponent of $a_1$ will be 2, and the remainder modulo 9 of $b_n$ will be 7 for every $n \geq 3$.

Hence, the answer is 7.