SOME INEQUALITIES

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Introduction. These are a few useful inequalities. Most of them are presented in two versions: in sum form and in integral form. More generally they can be viewed as inequalities involving vectors, the sum version applies to vectors in $\mathbb{R}^n$ and the integral version applies to spaces of functions.

First a few notations and definitions.

Absolute value. The absolute value of $x$ is represented $|x|$.

Norm. Boldface letters line $u$ and $v$ represent vectors. Their scalar product is represented $u \cdot v$. In $\mathbb{R}^n$ the scalar product of $u = (a_1, \ldots, a_n)$ and $v = (b_1, \ldots, b_n)$ is

$$ u \cdot v = \sum_{i=1}^{n} a_i b_i. $$

For functions $f, g : [a, b] \to \mathbb{R}$ their scalar product is

$$ \int_{a}^{b} f(x) g(x) \, dx. $$

The $p$-norm of $u$ is represented $\|u\|_p$. If $u = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, its $p$-norm is:

$$ \|u\|_p = \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p}. $$

For functions $f : [a, b] \to \mathbb{R}$ the $p$-norm is defined:

$$ \|f\|_p = \left( \int_{a}^{b} |f(x)|^p \, dx \right)^{1/p}. $$

For $p = 2$ the norm is called Euclidean.
Convexity. A function \( f : (a, b) \rightarrow \mathbb{R} \) is said to be *convex* if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

for every \( x, y \in (a, b) \), \( 0 \leq \lambda \leq 1 \). Graphically, the condition is that for \( x < t < y \) the point \((t, f(t))\) should lie below or on the line connecting the points \((x, f(x))\) and \((y, f(y))\).

![Figure 1. Convex function.](image)

Inequalities

1. *Arithmetic-Geometric Mean Inequality.* (Consequence of convexity of \( e^x \) and Jensen’s inequality.) The geometric mean of positive numbers is not greater than their arithmetic mean, i.e., if \( a_1, a_2, \ldots, a_n > 0 \), then

\[
\left( \prod_{i=1}^{n} a_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i.
\]

Equality happens only for \( a_1 = \cdots = a_n \). (See also the power means inequality.)

2. *Arithmetic-Harmonic Mean Inequality.* The harmonic mean of positive numbers is not greater than their arithmetic mean, i.e., if \( a_1, a_2, \ldots, a_n > 0 \), then

\[
\frac{n}{\sum_{i=1}^{n} 1/a_i} \leq \frac{1}{n} \sum_{i=1}^{n} a_i.
\]

Equality happens only for \( a_1 = \cdots = a_n \).

This is a particular case of the Power Means Inequality.
3. **Cauchy.** (Hölder for $p = q = 2$.)

$$ |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2. $$

$$ \left| \sum_{i=1}^{n} a_i b_i \right|^2 \leq \left| \sum_{i=1}^{n} a_i^2 \right| \left| \sum_{i=1}^{n} b_i^2 \right|. $$

$$ \left| \int_{a}^{b} f(x) g(x) \, dx \right|^2 \leq \left( \int_{a}^{b} |f(x)|^2 \, dx \right) \left( \int_{a}^{b} |g(x)|^2 \, dx \right). $$

4. **Chebyshev.** Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be sequences of real numbers which are monotonic in the same direction (we have $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, or we could reverse all inequalities.) Then

$$ \frac{1}{n} \sum_{i=1}^{n} a_i b_i \geq \left( \frac{1}{n} \sum_{i=1}^{n} a_i \right) \left( \frac{1}{n} \sum_{i=1}^{n} b_i \right). $$

Note that $\text{LHS} - \text{RHS} = \frac{1}{2n^2} \sum_{i,j} (a_i - a_j)(b_i - b_j) \geq 0$.

5. **Geometric-Harmonic Mean Inequality.** The harmonic mean of positive numbers is not greater than their geometric mean, i.e., if $a_1, a_2, \ldots, a_n > 0$, then

$$ \sum_{i=1}^{n} \frac{n}{a_i} \leq \left( \prod_{i=1}^{n} a_i \right)^{1/n}. $$

Equality happens only for $a_1 = \cdots = a_n$.

This is a particular case of the Power Means Inequality.

6. **Hölder.** If $p > 1$ and $1/p + 1/q = 1$ then

$$ |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|_p \|\mathbf{v}\|_q. $$

$$ \left| \sum_{i=1}^{n} a_i b_i \right| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/q}. $$

$$ \left| \int_{a}^{b} f(x) g(x) \, dx \right| \leq \left( \int_{a}^{b} |f(x)|^p \, dx \right)^{1/p} \left( \int_{a}^{b} |g(x)|^q \, dx \right)^{1/q}. $$
7. **Jensen.** If \( \varphi \) is convex on \((a, b)\), \( x_1, x_2, \ldots, x_n \in (a, b) \), \( \lambda_i \geq 0 \) \((i = 1, 2, \ldots, n)\), \( \sum_{i=1}^{n} \lambda_i = 1 \), then
\[
\varphi \left( \sum_{i=1}^{n} \lambda_i x_i \right) \leq \sum_{i=1}^{n} \lambda_i \varphi(x_i).
\]

8. **MacLaurin’s Inequalities.** Let \( e_k \) be the \( k \)th degree elementary symmetric polynomial in \( n \) variables:
\[
e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2}\cdots x_{i_k}.
\]
Given positive numbers \( a_1, a_2, \ldots, a_n \), let \( S_k = e_k(a_1, a_2, \ldots, a_n)/(n^k) \) be the averages of the elementary symmetric sums of the \( a_i \).
Then
\[
S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \cdots \geq \sqrt[n]{S_n},
\]
with equality if and only if all the \( a_i \) are equal. (See [3].) (See also Newton’s Inequalities.)

9. **Minkowski.** If \( p > 1 \) then
\[
\|u + v\|_p \leq \|u\|_p + \|v\|_p,
\]
\[
\left( \sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/p},
\]
\[
\left( \int_{a}^{b} |f(x) + g(x)|^p \, dx \right)^{1/p} \leq \left( \int_{a}^{b} |f(x)|^p \, dx \right)^{1/p} + \left( \int_{a}^{b} |g(x)|^p \, dx \right)^{1/p}.
\]
Equality holds iff \( u \) and \( v \) are proportional.

10. **Muirhead’s Inequality.** Given real numbers \( a_1 \geq \cdots \geq a_n \), and \( b_1 \geq \cdots \geq b_n \), assume that \( \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \) for \( i = 1, \ldots, n - 1 \), and \( \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \). Then for any nonnegative real numbers \( x_1, \ldots, x_n \), we have
\[
\sum_{\sigma} x_{\sigma_1}^{a_1} \cdots x_{\sigma_n}^{a_n} \leq \sum_{\sigma} x_{\sigma_1}^{b_1} \cdots x_{\sigma_n}^{b_n},
\]
where the sums extend over all permutations \( \sigma \) of \( \{1, \ldots, n\} \) (see theorem 2.18 in [1].)
11. **Newton’s Inequalities.** Let $e_k$ be the $k$th degree elementary symmetric polynomial in $n$ variables:

$$
 e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} .
$$

Given positive numbers $a_1, a_2, \ldots, a_n$, let $S_k = e_k(a_1, a_2, \ldots, a_n)/(n\choose k)$ be the averages of the elementary symmetric sums of the $a_i$ for $k \geq 1$, and $S_0 = 1$. Then (for $k = 1, 2, \ldots, n - 1$):

$$
 S_{k-1} S_{k+1} \leq S_k^2 ,
$$

with equality if and only if all the $a_i$ are equal. (See also MacLaurin’s Inequalities).

12. **Norm Monotonicity.** If $a_i > 0$ ($i = 1, 2, \ldots, n$), $s > t > 0$, then

$$
 \left( \sum_{i=1}^{n} a_i^s \right)^{1/s} \leq \left( \sum_{i=1}^{n} a_i^t \right)^{1/t} ,
$$

i.e., if $s > t > 0$, then $\|u\|_s \leq \|u\|_t$.

13. **Power Means Inequality.** Let $r$ be a non-zero real number. We define the $r$-mean or $r$th power mean of non-negative numbers $a_1, \ldots, a_n$ as follows:

$$
 M^r(a_1, \ldots, a_n) = \left( \frac{1}{n} \sum_{i=1}^{n} a_i^r \right)^{1/r} .
$$

If $r < 0$, and $a_k = 0$ for some $k$, we define $M^r(a_1, \ldots, a_n) = 0$.

The ordinary arithmetic mean is $M^1$, $M^2$ is the quadratic mean, $M^{-1}$ is the harmonic mean. Furthermore we define the 0-mean to be equal to the geometric mean:

$$
 M^0(a_1, \ldots, a_n) = \left( \prod_{i=1}^{n} a_i \right)^{1/n} .
$$

Then for any real numbers $r, s$ such that $r < s$, the following inequality holds:

$$
 M^r(a_1, \ldots, a_n) \leq M^s(a_1, \ldots, a_n) .
$$

Equality holds if and only if $a_1 = \cdots = a_n$, or $s \leq 0$ and $a_k = 0$ for some $k$. (See weighted power means inequality).
14. **Power Means Sub/Superadditivity.** We use the definition of $r$-mean given in subsection 13. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be non-negative real numbers.

(1) If $r > 1$, then the $r$-mean is **subadditive**, i.e.:
\[ M^r(a_1 + b_1, \ldots, a_n + b_n) \leq M^r(a_1, \ldots, a_n) + M^r(b_1, \ldots, b_n). \]

(2) If $r < 1$, then the $r$-mean is **superadditive**, i.e.:
\[ M^r(a_1 + b_1, \ldots, a_n + b_n) \geq M^r(a_1, \ldots, a_n) + M^r(b_1, \ldots, b_n). \]

Equality holds if and only if $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional, or $r \leq 0$ and $a_k = b_k = 0$ for some $k$.

15. **Radon’s Inequality.** For real numbers $p > 0$, $x_1, \ldots, x_n \geq 0$, $a_1, \ldots, a_n > 0$, the following inequality holds:
\[ \sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p}. \]

**Remark:** Radon’s Inequality follows from Hölder’s $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|_{p+1} \|\mathbf{v}\|_{q}$, with $\mathbf{u} = (x_1/a_1^{1/q}, \ldots, x_n/a_n^{1/q})$, $\mathbf{v} = (a_1^{1/q}, \ldots, a_n^{1/q})$, $1/p + 1/q = 1$.

16. **Rearrangement Inequality.** For every choice of real numbers $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$, and any permutation $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ of $x_1, \ldots, x_n$, we have
\[ x_n y_1 + \cdots + x_1 y_n \leq x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \leq x_1 y_1 + \cdots + x_n y_n. \]

If the numbers are different, e.g., $x_1 < \cdots < x_n$ and $y_1 \lessdot \cdots \lessdot y_n$, then the lower bound is attained only for the permutation which reverses the order, i.e. $\sigma(i) = n - i + 1$, and the upper bound is attained only for the identity, i.e. $\sigma(i) = i$, for $i = 1, \ldots, n$.

17. **Schur.** If $x, y, z$ are positive real numbers and $k$ is a real number such that $k \geq 1$, then
\[ x^k(x - y)(x - z) + y^k(y - x)(y - z) + z^k(z - x)(z - y) \geq 0. \]

For $k = 1$ the inequality becomes
\[ x^3 + y^3 + z^3 + 3xyz \geq xy(x + y) + yz(y + z) + zx(z + x). \]
18. **Schwarz.** (Hölder with \( p = q = 2 \).)

\[
|u \cdot v| \leq \|u\|_2 \|v\|_2,
\]

\[
\left| \sum_{i=1}^{n} a_i b_i \right|^2 \leq \left( \sum_{i=1}^{n} |a_i|^2 \right) \left( \sum_{i=1}^{n} |b_i|^2 \right),
\]

\[
\left| \int_{a}^{b} f(x)g(x) \, dx \right|^2 \leq \left( \int_{a}^{b} |f(x)|^2 \, dx \right) \left( \int_{a}^{b} |g(x)|^2 \, dx \right).
\]

19. **Strong Mixing Variables Method.** We use the definition of \( r \)-mean given in subsection 13. Let \( F : I \subset \mathbb{R}^n \to \mathbb{R} \) be a symmetric, continuous function satisfying the following: for all \( (x_1, x_2, \ldots, x_n) \in I \) such that \( 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \), \( F(x_1, x_2, \ldots, x_n) \geq F(t, x_2, \ldots, x_{n-1}, t) \), where \( t = M^r(x_1, x_n) \). Then:

\[
F(x_1, x_2, \ldots, x_n) \geq F(x, x, \ldots, x),
\]

where \( x = M^r(x_1, x_2, \ldots, x_n) \).

An analogous result holds replacing \( \geq \) with \( \leq \).

20. **Weighted Power Means Inequality.** Let \( w_1, \ldots, w_n \) be positive real numbers such that \( w_1 + \cdots + w_n = 1 \). Let \( r \) be a non-zero real number. We define the \( r \)th weighted power mean of non-negative numbers \( a_1, \ldots, a_n \) as follows:

\[
M_w^r(a_1, \ldots, a_n) = \left( \sum_{i=1}^{n} w_i a_i^r \right)^{1/r}.
\]

As \( r \to 0 \) the \( r \)th weighted power mean tends to:

\[
M_w^0(a_1, \ldots, a_n) = \left( \prod_{i=1}^{n} a_i^{w_i} \right).
\]

which we call 0th weighted power mean. If \( w_i = 1/n \) we get the ordinary \( r \)th power means.

Then for any real numbers \( r, s \) such that \( r < s \), the following inequality holds:

\[
M_w^r(a_1, \ldots, a_n) \leq M_w^s(a_1, \ldots, a_n).
\]

(If \( r, s \neq 0 \) note convexity of \( x^{s/r} \) and recall Jensen’s inequality.)
References

