MATH 483-1: PROBLEM SET 1

Due: Wednesday, October 8

In the homework [Ha] will always denote Hartshorne’s book “Algebraic Geometry”.

1. [Ha] Ch.I: 1.2, 1.3, 1.4, 1.6, 1.7, 1.9, 1.11 (challenge), 3.2, 3.10, 3.11, 3.12, 3.13

2. Let $X$ and $Y$ be algebraic sets in $\mathbb{A}^n$. Show that
   (i) If $X \subset Y$, then $I(Y) \subset I(X)$.
   (ii) $I(X \cup Y) = I(X) \cap I(Y)$.
   (iii) $I(X \cap Y) = \text{rad}(I(X) + I(Y))$.

   Give an example of algebraic sets $X$ and $Y$ such that $I(X \cap Y) \neq I(X) + I(Y)$.

3. Prove (by interpreting both sides geometrically) that there are three points $p, q, r$ in the affine plane $\mathbb{A}^2$ such that
   $$\text{rad}(x^2 - 2xy^4 + y^6, y^3 - y) = I(\{p\}) \cap I(\{q\}) \cap I(\{r\})$$

   Explain, or find at least a heuristic reason, why $(x^2 - 2xy^4 + y^6, y^3 - y)$ is not a radical ideal (this hints towards schemes).

4. Let $Z$ be the curve given by $Y^2 = X(X-1)(X-\lambda)$ in $\mathbb{A}^2$, where $\lambda \in k \setminus \{0,1\}$ (this is the equation of an elliptic curve). Prove that $Z$ is irreducible.

5. Let $U$ be a nonempty open subset in the affine variety $X$. Then
   $$\dim U = \dim X.$$

6. Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be affine algebraic sets. Show that $X \times Y \subset \mathbb{A}^{m+n}$ is an algebraic set with coordinate ring $A(X \times Y) \cong A(X) \otimes_k A(Y)$. Show that $X \times Y$ is irreducible if $X$ and $Y$ are. Show that $\dim X \times Y = \dim X + \dim Y$.

7. Let $n \geq 2$ and $P \in \mathbb{A}^n$ a point. Show that $X = \mathbb{A}^n \setminus \{P\}$ is not affine. [Hint: What are the regular functions on $X$? Compare with the theorem on removable singularities in complex analysis.]

8. (i) Let $X \subset \mathbb{A}^n$ be an affine variety, and let $f \in A(X)$. Show that the distinguished open set $X_f$ is isomorphic to an affine subvariety in $\mathbb{A}^{n+1}$, and $A(X_f) \cong A(X)_f$.

   (ii) Show that as $f$ varies over $A(X)$, the open sets $X_f$ form a basis for the topology of $X$.  

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