MATH 483-1: PROBLEM SET 3

Due Wednesday, November 5

For the material in the first four problems, you can consult for instance “Algebraic Geometry” by J. Harris, Lectures 4, 6, and 11, or the section on Grassmannians in “Principles of Algebraic Geometry” by Griffiths-Harris. There are of course many other references on Grassmannians.

(1) Let $G(r,V)$ be the Grassmannian of $r$-dimensional linear subspaces in a vector space $V$ (equivalently, the set of $(r-1)$-dimensional linear subspaces in $P(V)$, for which I will use the notation $G(r-1,P(V))$ – if $V$ has dimension $n$, and we only care about this, we also use the notation $G(r,n)$, and $G(r-1,n-1)$, respectively). Give a brief sketch of the proof of the following facts (and think about the full details for yourselves):

(a) $G(r,V)$ has a natural structure of variety (so in particular is irreducible), and admits an embedding with closed image into $P(\wedge^r V)$ (the Plücker embedding), showing that $G(r,V)$ is a projective variety.

(b) The incidence variety (or universal $(r-1)$-plane)
$$\{([\Lambda],[x]) \in G(r-1,P(V)) \times P(V) \mid x \in \Lambda\}$$
is closed.

(2) Let $f : X \to Y$ be a surjective morphism of projective closed sets, with $Y$ irreducible and with all the fibers of $f$ irreducible of the same dimension. Then $X$ is irreducible.

(3) Let $X \subset P^n$ be a projective variety, and consider an integer $k \leq n - \dim X$. Define the following subset of the (projective) Grassmannian
$$I_k(X) := \{W \mid W \cap X \neq \emptyset\} \subset G(k,n).$$
Show that $I_k(X)$ is an irreducible subvariety of $G(k,n)$ of dimension $k(n-k) + \dim X$. This is called the variety of incident $k$-planes. (Hint: use the usual incidence correspondence in $G(k,n) \times P^n$ discussed in class and in the exercise above, and the fact (which you need to show) that a general $k$ plane in $I_k(X)$ intersects $X$ only in a finite number of points.)

(4) Fix a line $l_0$ in $P^3$, and consider $I_1(l_0) \subset G(1,3)$ as in the previous problem. Show that the projection from the incidence correspondence to $I_1(l_0)$ is a birational morphism whose exceptional locus has codimension $> 1$ (unlike a blow-up).
(5) Let $X$ and $Y$ be disjoint subvarieties in $\mathbb{P}^n$. Show that the union $J(X,Y)$ of the lines connecting a point of $X$ with a point of $Y$ is a projective subvariety of $\mathbb{P}^n$ (called the *join* of $X$ and $Y$). Show that the dimension of $J(X,Y)$ is $\dim X + \dim Y + 1$.

(6) [Ha] Ch.II: 2.4, 2.7, 2.8

(7) Let $X$ be a scheme. Show that for every irreducible closed subset $Y \subset X$ there is a unique point $\eta_Y \in X$ such that $Y$ is the closure of $\{\eta_Y\}$. The point $\eta_Y$ is called the *generic point* of $Y$.

(8) In the situation of the previous exercise, we write $\mathcal{O}_{X,Y}$ for the stalk $\mathcal{O}_{X,\eta_Y}$. Show that $\mathcal{O}_{X,Y}$ is “the ring of rational functions on $X$ that are regular at the general point of $Y$”, i.e. it is isomorphic to the ring of equivalence classes of pairs $(U, \phi)$, where $U \subset X$ is open with $U \cap Y \neq \emptyset$ and $\phi \in \mathcal{O}_X(U)$, and where two such pairs $(U, \phi)$ and $(U', \phi')$ are equivalent if there is an open subset $V \subset U \cap U'$ with $V \cap Y \neq \emptyset$ such that $\phi|_V = \phi'|_V$. (In particular, if $X$ is a scheme that is a variety, then $\mathcal{O}_{X,\eta_X}$ is the function field of $X$ as defined earlier. Hence the stalks of the structure sheaf of a scheme generalize both the concepts of the local ring and function field of a variety.)

(9) Let $k$ be a field and let $\mathbb{A}^3 = \mathbb{A}^3_k$. Let $L \subset \mathbb{A}^3$ be the line $\{x_1 = x_2 = 0\}$, and for $\alpha \in k$ let $L_\alpha$ be the line $\{x_3 = 0, x_1 = \alpha\}$. We consider the subschemes $L \cup L_\alpha$ as $\alpha$ varies, and in particular look at the limit when $\alpha = 0$. To do this, let $X$ be the algebraic set

$$X := \{(x_0, x_1, x_2, x_3) \mid (x_1, x_2, x_3) \in L \cup L_{x_0} \subset \mathbb{A}^3\} \subset \mathbb{A}^1 \times \mathbb{A}^3,$$

and for $\alpha \in k$ let $X_\alpha$ be the scheme $\text{Spec } k[x_0, x_1, x_2, x_3]/(I(X) + (x_0 - \alpha))$. Obviously $X_\alpha \subset \{\alpha\} \times \mathbb{A}^3 \cong \mathbb{A}^3$. Describe the scheme $X_\alpha$, in particular for $\alpha = 0$. Is $X_0$ just the union $L \cup L_0$? Is it reduced? Is it contained in the plane $\{x_1 = 0\} \subset \mathbb{A}^3$?