A. STANDARD VANISHING THEOREMS AND GENERIC VANISHING THEOREMS

1. Vanishing theorems. Let $X$ be a smooth projective variety over $\mathbb{C}$. If $L$ is an ample line bundle on $X$, then the celebrated Kodaira Vanishing theorem says that

$$H^i(X, \omega_X \otimes L) = 0 \text{ for all } i > 0.$$  

This of course has myriads of applications.

Let’s recall also that Kodaira vanishing has important generalizations, which will be used below. First, some definitions:

Definition 1.1. Let $X$ be a projective variety, and $L$ a line bundle on $X$.

1. $L$ is nef if $L \cdot C \geq 0$ for any irreducible curve $C \subset X$.

2. $L$ is big if there exist constants $A, B > 0$ such that

$$A \cdot m^n \leq h^0(X, L^\otimes m) \leq B \cdot m^n \text{ for } m \gg 0.$$  

Equivalently,

$$\text{vol}(L) := \lim_{m \to \infty} \frac{h^0(X, L^\otimes m)}{m^n/n!} > 0.$$  

This means essentially that the number of sections of $L^\otimes m$ grows as fast as possible with $m$.

Example 1.2. (Exercise.) If $L$ is ample, then $L$ is clearly both nef and big. More generally, our main example will be the following: let $f : X \to Y$ be a generically finite morphism, and $M$ an ample line bundle on $Y$. Then $L = f^* M$ is nef and big on $X$ (though definitely not ample if $f$ is not finite).

Exercise 1. (1) (You might need to read this somewhere, e.g. [KM98] Proposition 2.61) Let $L$ be a nef line bundle. Then $L$ is big if and only if $L^{\dim X} > 0$.

(2) Show that if $L$ is a nef line bundle and $M$ is a nef and big line bundle, then $L \otimes M$ is nef and big.

A well-known generalization of Kodaira Vanishing is the following:

Theorem 1.3 (Kawamata-Viehweg Vanishing). Let $X$ be a smooth projective complex variety, and $L$ a big and nef line bundle on $X$. Then

$$H^i(X, \omega_X \otimes L) = 0 \text{ for all } i > 0.$$  

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1The second inequality holds for any $L$ and for all $m$, so the content of the definition is of course the first one.
There is a further, and deeper, generalization of this to \(\mathbb{Q}\)-divisors, which I will not describe here. Here is however yet another generalization that we will use in these lectures.

**Theorem 1.4** (Kollár [Kol86a], [Kol86b]). Let \( f : X \to Y \) be a morphism of complex projective varieties, with \( X \) smooth, and let \( L \) be a big and nef line bundle on \( Y \). Then:

1. \( R^if_*\omega_X \) is torsion-free for all \( i \).
2. In \( D(Y) = D^b(\text{Coh}(Y)) \) one has
   \[
   Rf_*\omega_X \simeq \bigoplus_i R^if_*\omega_X[-i].
   \]
   In particular, the Leray spectral sequence degenerates to give, for any line bundle \( N \in \text{Pic}(Y) \),
   \[
   H^i(X, \omega_X \otimes f^*N) \simeq \bigoplus_{j=0}^i H^{i-j}(Y, R^jf_*\omega_X \otimes N).
   \]
3. \( H^i(Y, R^j f_*\omega_X \otimes L) = 0 \) for all \( i > 0 \) and all \( j \).

**Remark.** The Theorem can be extended to the case \( \omega_X \otimes N \), with \( N \) a torsion line bundle, in place of \( \omega_X \). This follows from the statement above, given that one can find an étale cover \( \pi : Y \to X \) such that \( \pi^*N \simeq \mathcal{O}_Y \); exercise!

2. **Generic vanishing theorems.** We see that there are good vanishing theorems in the presence of various forms of positivity. However, one hopes that some useful results might hold even under weaker positivity hypotheses. So a natural question arises: are there any modified vanishing theorems involving just \( \omega_X \), or more generally \( \omega_X \otimes L \), with say \( L \) a nef line bundle?

**Example 2.1.** (1) Let \( C \) be a smooth projective curve of genus \( g \geq 1 \). Then we have \( H^1(C, \omega_C) \cong \mathbb{C} \), but also \( H^1(C, \omega_C \otimes \alpha) = 0 \) for any \( \alpha \in \text{Pic}^0(C) - \{0\} \). Hence the higher cohomology of a general (and in fact any nontrivial) twist of the canonical line bundle by a topologically trivial line bundle vanishes. This is an instance of “generic vanishing”.

(1') In fact, due to Serre duality, the phenomenon in the example above holds in arbitrary dimension: if \( X \) is smooth projective of dimension \( n \), for \( L \in \text{Pic}^0(X) \),
   \[
   H^n(X, \omega_X \otimes L) \neq 0 \iff L \simeq \mathcal{O}_X.
   \]

(2) The picture becomes more interesting in higher dimension. I will exemplify with the case of elliptic surfaces. Let
   \[
   X = S \xrightarrow{L} C
   \]
   be an elliptic surface over a curve of genus \( g \geq 2 \). We’ve seen that the situation for \( H^2 \) is completely determined. However, the picture for \( H^1 \) is interesting; for \( L \in \text{Pic}^0(C) \), one can use
   \[
   H^1(S, \omega_S \otimes f^*L) \simeq H^1(C, f_*\omega_S \otimes L) \oplus H^0(C, \omega_C \otimes L) \neq 0.
   \]
This follows from Theorem 1.4 above, plus another result of Kollár saying that $R^1 f_* \omega_S \simeq \omega_C$.

The situation is split into two cases; for all the facts on elliptic surfaces used here one can consult [Bea96] Ch.IX.

• $f$ non-isotrivial: This means that the smooth fibers of $f$ vary in moduli. In this case it is well known that $q(S) = g$, and so

\[ \text{Pic}^0(S) = f^* \text{Pic}^0(C) = \{ M \mid H^1(S, \omega_S \otimes M) \neq 0 \}. \]

Note that $\text{Alb}(S) = J(C)$, and the Albanese map of $S$ is the composition

\[ S \xrightarrow{f} C \hookrightarrow J(C), \]

so it contracts the fibers of $f$. There is no vanishing of $H^1$ whatsoever for the twists of $\omega_S$.

• $f$ isotrivial: This means that there exists a nonempty open set $U \subseteq C$ such that $F_x \simeq F_y$ for every $x, y \in U$. In this case $q(S) = g + 1$. Denoting by $F$ the general fiber of $f$, for the Albanese map of $S$ one has the following diagram

\[
\begin{array}{ccc}
F & \xrightarrow{=} & S \\
\downarrow & & \downarrow \text{alb}_S \\
F' & \xrightarrow{=} & \text{Alb}(S)
\end{array}
\]

\[
\begin{array}{ccc}
& & C \\
\downarrow & & \downarrow \\
& & J(C)
\end{array}
\]

where $F \to F'$ is an isogeny onto an elliptic curve in $\text{Alb}(S)$. Therefore the Albanese map of $S$ is finite, and

\[ f^* \text{Pic}^0(C) \subseteq \{ M \mid H^1(S, \omega_S \otimes M) \neq 0 \} \subseteq \text{Pic}^0(S) \]

is a divisorial component in the non-vanishing locus which is an abelian subvariety. (We will see below that $H^1(S, \omega_S \otimes L) = 0$ for $L$ general.)

These examples are special cases of the following general phenomenon discovered by Green and Lazarsfeld. To state the result, we introduce the following notation:

\[ V^i(\omega_X) := \{ \alpha \in \text{Pic}^0(X) \mid H^i(X, \omega_X \otimes \alpha) \neq 0 \} \subseteq \text{Pic}^0(X). \]

By the semicontinuity theorem this is a closed subset for each $i$, called the $i$-th cohomological support locus of $\omega_X$. E.g. for a curve $C$ of genus $g \geq 2$ we have $V^0(\omega_C) = \text{Pic}^0(C)$ and $V^1(\omega_C) = \{0\}$.

Exercise 2. Compute the cohomological support loci for the two types of elliptic surfaces in the example above. (You might want to consult [Pop12b] §2.)

**Theorem 2.2 (Dimension Theorem, Green-Lazarsfeld [GL87]).** Consider the Albanese map $a : X \to \text{Alb}(X)$. Then

\[ \text{codim } V^i(\omega_X) \geq i - \dim X + \dim a(X) \] for all $i$.

**Theorem 2.3 (Linearity Theorem, Green-Lazarsfeld [GL91]; Arapura [Ara92], Simpson [Sim93]).** The irreducible components of each $V^i(\omega_X)$ are torsion translates of abelian subvarieties of $\text{Pic}^0(X)$. 
Theorem 2.4 (Fibrations Theorem, Green-Lazarsfeld [GL91]). Let $W$ be a positive dimensional irreducible component of $V^i(\omega_X)$ for some $i$. Then there exists a morphism $f : X \to Y$, with $Y$ normal, of maximal Albanese dimension, and $\dim Y \leq n - i$, such that

$$W \subseteq \tau + f^* \text{Pic}^0(Y), \quad \tau \in \text{Pic}^0(X).$$

The main example for understanding the Dimension Theorem is the following.

Definition 2.5. We say that $X$ is of maximal Albanese dimension if $a = \text{alb}_X$ is generically finite onto its image, or in other words $\dim X = \dim a(X)$. Some familiar examples of such varieties are: subvarieties of tori (or their smooth models), isotrivial elliptic surfaces, symmetric products of curves.

Note that in the case of maximal Albanese dimension, the Dimension Theorem implies in particular that $V^i(\omega_X)$ are proper subsets for $i > 0$, which gives

Corollary 2.6. If $X$ is of maximal Albanese dimension, then

$$H^i(X, \omega_X \otimes L) = 0, \quad \forall \ i > 0, \quad \forall \ L \in \text{Pic}^0(X) \text{ general}.$$  

This leads to the simplest instance in which generic vanishing theorems can be used in a similar way to Kodaira-type vanishing theorems.

Corollary 2.7. If $X$ is of maximal Albanese dimension, then $\chi(\omega_X) \geq 0$.

Proof. Note that if $P$ is a Poincaré line bundle on $X \times \text{Pic}^0(X)$, then $p_Y^* \omega_X \otimes P$ is a flat family of line bundles one $X$, whose restriction to $X \times \{L\}$ is $\omega_X \otimes L$. It follows that for every $L \in \text{Pic}^0(X)$, by the deformation invariance of the Euler characteristic we have

$$\chi(\omega_X) = \chi(\omega_X \otimes L).$$

On the other hand, if $L$ is general, by the Corollary above we have that

$$\chi(\omega_X \otimes L) = h^0(X, \omega_X \otimes L) \geq 0.$$  

Remark (Generalizations). Theorem 7.3 was extended to all sheaves of the form $R^i f_* \omega_Y$, with $f : Y \to X$ a surjective morphism from a smooth projective variety $Y$ by Hacon [Hac04]. Furthermore, it was extended to all line bundles of the form $\omega_X \otimes L$ with $L \text{ nef}$ (this is what we will prove below), and similarly to $R^i f_* \omega_Y \otimes L$ with $f$ as above and $L \text{ nef}$, in [PP11].

3. A proof of the Dimension Theorem. I will follow an approach introduced in [Hac04] and expanded in [PP11]. It is worth however to exemplify part of the method with a well-known and a priori seemingly unrelated result.

Theorem 3.1 (Grauert-Riemenschneider Vanishing). Let $f : X \to Y$ be a generically finite morphism of complex projective varieties, with $X$ smooth. Then

$$R^i f_* \omega_X = 0 \text{ for all } i > 0.$$  

First a few remarks: the theorem can be seen as a special case of Kollár’s Theorem 1.4(1), which actually gives something more general: if \( f : X \to Y \) is a surjective morphism of complex projective varieties, with \( X \) smooth, then
\[
R^i f_* \omega_X = 0 \quad \text{for all } i > \dim X - \dim Y.
\]
Indeed, for \( i > \dim X - \dim Y \) the sheaf \( R^i f_* \omega_X \) must be both torsion (by base-change) and torsion-free, and hence 0. Moreover, the statement is in fact local: it holds when \( X \) and \( Y \) are not necessarily projective, by using suitable compactifications to reduce to the projective case. I will however give here only a direct proof of what is stated in the theorem, as this is what is of interest below.

**Proof.** So let’s assume that \( f \) is generically finite. Pick a very positive ample line bundle \( L \) on \( Y \) (i.e. a sufficiently high power of another ample line bundle; we will use the notation \( L \gg 0 \)). Consider the Leray spectral sequence
\[
E^{p,q}_2 = H^p(Y, R^q f_* \omega_X \otimes L) = H^{p+q}(X, \omega_X \otimes f^* L).
\]
Since \( L \) is chosen very positive, Serre Vanishing implies that
\[
E^{p,q}_2 = H^p(Y, R^q f_* \omega_X \otimes L) = 0 \quad \forall q \text{ and } \forall p > 0.
\]
This means that the spectral sequence degenerates, and so for all \( i \) one has
\[
H^i(Y, R^i f_* \omega_X \otimes L) \simeq H^i(X, \omega_X \otimes f^* L).
\]
But by Example 1.2, \( f^* L \) is big and nef, and so Kawamata-Viehweg vanishing says that
\[
H^i(X, \omega_X \otimes f^* L) = 0 \quad \forall i > 0.
\]
We deduce that for all \( i > 0 \) and all \( L \gg 0 \)
\[
H^0(Y, R^i f_* \omega_X \otimes L) = 0.
\]
But again by Serre’s theorem this is impossible unless \( R^i f_* \omega_X = 0 \). \( \square \)

If you analyze the argument a bit, what we have really proved here is the following: if \( f : X \to Y \) is a morphism of projective varieties and \( \mathcal{F} \) is a coherent sheaf on \( X \), then
\[
R^i f_* \mathcal{F} = 0 \iff H^i(X, \mathcal{F} \otimes f^* L) = 0 \quad \forall L \gg 0 \text{ on } Y.
\]

In this form, the picture can be extended to **correspondences**, or **integral functors**. Note that if we denote by \( \Gamma_f \subset X \times Y \) the graph of \( f \), then a simple calculation shows that for a sheaf \( \mathcal{F} \) on \( X \)
\[
R f_* \mathcal{F} = R_{pY*}(p_X^* \mathcal{F} \otimes \mathcal{O}_{\Gamma_f}).
\]
Using this approach, the picture above extends in full generality (however, no worries if you are not familiar with derived categories, since here we will only be using the case when \( P \) is a line bundle). We will use the notation \( D(X) := D^b(\text{Coh}(X)) \), the bounded derived category of coherent sheaves on \( X \).

**Theorem 3.2.** Let \( X \) and \( Y \) be smooth projective varieties, and \( P \) an object in \( D(X \times Y) \). Consider the integral functors
\[
R \Phi_P : D(X) \to D(Y), \quad R \Phi_P A := R_{p_2*}(p^*_1 A \otimes P).
\]
and
\[ R\Psi_P : D(Y) \to D(X), \quad R\Psi_P B := R\pi_1_*(p_2^*B \otimes P). \]
Then, for any object \( A \) in \( D(X) \) and any \( i \), one has
\[ R^i\Phi_P A = 0 \iff H^i(X, A \otimes R\Psi_PL) = 0 \]
for every sufficiently positive ample line bundle \( L \) on \( Y \).

Modulo a few technicalities, the proof is very similar to that of the Grauert-Riemenschneider theorem (see the comment following it), so I will skip it here. Let’s specialize this to our situation of interest, and add a few more statements. We will use the notation \( R\Delta \mathcal{F} := R\mathcal{H}om(\mathcal{F}, \omega_X) \).

**Theorem 3.3.** Let \( X \) and \( Y \) be smooth projective varieties of dimensions \( d \) and \( g \) respectively, \( P \) a vector bundle on \( X \times Y \) defining the functor \( R\Phi_P, \mathcal{F} \) a sheaf on \( X \) and \( k \geq 0 \) an integer. The following conditions are equivalent:

1. \( \text{codim Supp } R^i\Phi_P \mathcal{F} \geq i - k \) for all \( i \geq 0 \).
2. \( R^i\Phi_P (R\Delta \mathcal{F}) = 0 \) for all \( i \notin [d - k, d] \).
3. \( H^i(X, \mathcal{F} \otimes R\Psi_P(L^{-1})) = 0 \) for all \( i > k \) and all \( L \gg 0 \) on \( Y \).

**Proof.** I will explain only the case \( k = 0 \) for simplicity, the general case is only notationally slightly more tedious. First, a key point to note is that a standard application of Grothendieck Duality gives the formula
\[ (3.4) \quad (R\Phi_P \mathcal{F})^\vee \cong R\Phi_P (R\Delta \mathcal{F})[d], \]
where \((\cdot)^\vee\) stands for the usual dual \( R\mathcal{H}om(\cdot, \mathcal{O}_X)\); see [PP11] Lemma 2.2.

(2) \( \implies \) (1): Denote \( \mathcal{G} := R^d\Phi_P (R\Delta \mathcal{F}) \), the only \( R^d\Phi_P (R\Delta \mathcal{F}) \) which is non-zero. Dualizing (3.4) one more time, for each \( i \) we obtain
\[ R^i\Phi_P \mathcal{F} \cong \mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_Y). \]

We then apply Lemma 5.1 below.

(1) \( \implies \) (2): I will skip this, since it is a bit more tedious and is not used in these notes; see [Pop12a] Corollary 3.5 for a proof.

(2) \( \iff \) (3): This is a consequence of Theorem 3.2, via duality. Indeed, by applying Grothendieck-Serre duality to the cohomology groups in (3), the vanishing there is equivalent to
\[ H^{d-i}(X, R\Delta \mathcal{F} \otimes R\Psi_P(L^{-1})^\vee) = 0 \quad \text{for all } i > 0. \]
Also, applying yet again the formula in (3.4), this time for the functor \( R\Psi_P \), this is in turn equivalent to
\[ H^i(X, R\Delta \mathcal{F} \otimes R\Psi_P, L) = 0 \quad \text{for all } i < d. \]
The result follows then immediately from Theorem 3.2. \( \Box \)

We are finally ready to prove the Dimension Theorem; in fact, following Hacon’s approach, we will prove its extension to line bundles of the form \( \omega_X \otimes L \) with \( L \) nef, as in [PP11].
Theorem 3.5. Let $X$ be a smooth projective variety, $a : X \to A = \text{Alb}(X)$ its Albanese map, and $L$ a nef line bundle on $X$. Then, for all $i$, $\text{codim } V^i(\omega_X \otimes L) \geq i - \dim X + \dim a(X)$.

Proof. We apply the equivalence between (1) and (3) in Theorem 3.3, with $k = \dim X - \dim a(X)$. Again, I will explain only the case $k = 0$ for simplicity; the general case requires only minor changes. According to the theorem, what we need to show is the vanishing

$$H^i(X, \omega_X \otimes L \otimes R\Psi_P(M^{-1})) = 0, \text{ for all } i > 0,$$

for any sufficiently positive ample line bundle $M$ on $\hat{A}$.

We can also consider the integral transform of $M$ with respect to a Poincaré line bundle $P$ on $A \times \hat{A}$. If we choose $P$ to be $(a \times \text{id})^*P$ via the diagram

$$\begin{array}{ccc}
X \times \hat{A} & \xrightarrow{a \times \text{id}} & A \times \hat{A} \\
p_X \downarrow & & \downarrow p_A \\
X & \to & A
\end{array}$$

then it is not hard to see (exercise!) that

$$R\Psi_P(M^{-1}) \simeq a^*R\Psi_P(M^{-1}).$$

(By the base change theorem, they are in fact both locally free sheaves placed in degree $q(X) = \dim A$.)

On the other hand, by Mukai’s Lemma 6.3 below, we have that

$$\phi_M^*R\Psi_P(M^{-1}) \simeq H^0(M) \otimes M^{-1},$$

where $\phi_M : \hat{A} \to A$ is the standard isogeny induced by $M$. We consider then the fiber product $X' := X \times_A \hat{A}$ induced by $a$ and $\phi_M$:

$$\begin{array}{ccc}
X' & \xrightarrow{\psi} & X \\
\downarrow b & & \downarrow a \\
\hat{A} & \xrightarrow{\phi_M} & A
\end{array}$$

It follows that

$$\psi^*R\Psi_P(M^{-1}) \simeq \psi^*a^*R\Psi_P(M^{-1}) \simeq b^*(H^0(M) \otimes M) \simeq H^0(M) \otimes b^*M. \tag{3.7}$$

Recall that we want to prove the vanishing in (3.6). Since $\psi$, like $\phi_M$, is étale, it is enough to prove this after pull-back to $X'$, so we need

$$H^i(X', \omega_{X'} \otimes \psi^*L \otimes \psi^*R\Psi_P(M^{-1})) = 0, \text{ for all } i > 0.$$ But by (3.7) we see that this amounts to having

$$H^i(X', \omega_{X'} \otimes \psi^*L \otimes b^*M) = 0, \text{ for all } i > 0.$$ However since $L$ is nef, so is any pullback, so $\psi^*L$ is nef. On the other hand $b^*M$ is the pullback of an ample line bundle by a generically finite map, so by Example 1.2 it is nef and big. Exercise 1 shows then that $\psi^*L \otimes b^*M$ is nef and big, so this last vanishing is a consequence of the Kawamata-Viehweg vanishing theorem. □
Remark (Transform of $\Theta_X$). It is important to draw some conclusions when we specialize to the case $\mathcal{F} = \omega_X$, $Y = \text{Pic}^0(X)$ and $P$ a Poincaré bundle in Theorem 3.3. Since we know that (1) and (3) hold for $\omega_X$, (2) must hold as well. Now by definition $R\Delta \omega_X \simeq \Theta_X$, so we obtain that

$$R^i\Phi_P \Theta_X = 0 \text{ for } i \notin [d-k,d].$$

Let’s specialize further to the case when $X$ is of maximal Albanese dimension. In this case $k = 0$, so we obtain that only $R^d\Phi_P \Theta_X \neq 0$. Let’s denote

$$\widehat{\Theta}_X := R^d\Phi_P \Theta_X.$$

It follows that $R\Phi_P \Theta_X = \Theta_X[-d]$, and so finally formula (3.4) gives

$$(3.8) \quad R^i\Phi_P \omega_X \simeq \mathcal{E}xt^i(\widehat{\Theta}_X, \Theta_\widehat{A}) \text{ for all } i.$$

This is crucial for applications.

4. Fourier-Mukai transform and GV-sheaves. We denote by $A$ an abelian variety of dimension $g$. Given a Poincaré bundle $P$ on $A \times \widehat{A}$, recall that we can consider the Fourier-Mukai functor

$$R\Phi_P : D(A) \rightarrow D(\widehat{A}), \mathcal{F} \mapsto Rp_{2*}(p_1^*\mathcal{F} \otimes P).$$

Here are some basic properties of this functor discovered by Mukai:

**Theorem 4.1** ([Muk81], Theorem 2.2). $R\Phi_P$ establishes an equivalence of derived categories. More precisely, there are natural isomorphisms

$$R\Phi_P \circ R\Psi_P \simeq (\text{1}_{\widehat{A}})^*[-g] \text{ and } R\Psi_P \circ R\Phi_P \simeq (\text{1}_A)^*[-g].$$

**Exercise 3.** Show that $R\Phi_P \Theta_A \simeq \Theta_{\{0_A\}}[-g]$. (Note that by the theorem above, this implies that if $R\Phi_P \mathcal{F} \simeq \Theta_{\{0_A\}}[-g]$, then $\mathcal{F} \simeq \Theta_A$, a fact that will be used below.)

Recall that an ample line bundle $M$ on $A$ induces an isogeny

$$\varphi_M : A \rightarrow \widehat{A}, \quad x \mapsto t_x^*M \otimes M^{-1},$$

where $t_x : A \rightarrow A$ denotes translation by $x$.

**Lemma 4.2** ([Muk81], Proposition 3.11). If $M$ is an ample line bundle on $A$, then

$$\varphi_M^*(R\Phi_P M) \simeq H^0(M) \otimes M^{-1} \quad (\simeq \bigoplus_{h^0(M)} M^{-1}).$$

**Definition 4.3.** A coherent sheaf $\mathcal{F}$ on $A$ is called a GV-sheaf if

$$\text{codim Supp } R^i\Phi_P \mathcal{F} \geq i \text{ for all } i \geq 0.$$

Here GV stands for Generic Vanishing; this is inspired by the Dimension Theorem, which says that $\omega_X$ is a GV-sheaf on a variety of maximal Albanese dimension.

**Lemma 4.4.** If $\mathcal{F}$ is a GV-sheaf on $A$, then

$$V^g(\mathcal{F}) \subseteq V^{g-1}(\mathcal{F}) \subseteq \cdots \subseteq V^0(\mathcal{F}).$$
Proof. To keep the argument simpler, I will assume that \( \mathcal{F} \) is locally free. By Serre duality we have for all \( i \) and all \( \alpha \in \text{Pic}^0(A) \) that
\[
H^i(A, \mathcal{F} \otimes \alpha) \cong H^{9-i}(A, \mathcal{F}^\vee \otimes \omega_X \otimes \alpha^{-1})^\vee.
\]
If this is 0, then by cohomology and base change we have that \( \text{Supp} \Phi^i_p(\mathcal{F}) \cap \text{Supp} \Phi^{i+1}_p(\mathcal{F}) = 0 \).

Lemma 4.5. For every \( m \in \mathbb{Z} \), the following conditions are equivalent:

(1) \( \text{codim} \text{Supp} R^i \Phi_p \mathcal{F} \geq i + m \) for all \( i \geq 0 \).

(2) \( \text{codim} V^i(\mathcal{F}) \geq i + m \) for all \( i \geq 0 \).

Proof. Since by cohomology and base change we have that \( \text{Supp} R^i \Phi_p \mathcal{F} \subseteq V^i(\mathcal{F}) \), it is enough to prove that (1) implies (2). The proof is by descending induction on \( i \). Note that clearly \( H^{9+1}(X, A \otimes L) = 0 \) for any \( L \in \text{Pic}^0(A) \), so by base change \( \text{Supp} R^9 \Phi_p \mathcal{F} = V^9(\mathcal{F}) \). The induction step is as follows: assume that there is a component \( W \) of \( V^i(\mathcal{F}) \) of codimension less than \( i + m \). Since (1) holds, the generic point of \( W \) cannot be contained in \( \text{Supp} R^i \Phi_p \mathcal{F} \) and so, again by base change, we have that \( W \subseteq V^{i+1}(\mathcal{F}) \). This implies that \( \text{codim} V^{i+1}(\mathcal{F}) < i + m \), which contradicts the inductive hypothesis.

Lemma 4.6. If \( \mathcal{F} \) is a GV-sheaf on \( A \) with \( \mathcal{F} \neq 0 \), then \( V^0(\mathcal{F}) \neq \emptyset \).

Proof. Assume that \( V^0(\mathcal{F}) = \emptyset \). Then according to Lemma 4.4, we have \( V^i(\mathcal{F}) = \emptyset \) for all \( i \). Given Lemma 4.5, this in turn implies \( R^i \Phi_p \mathcal{F} = 0 \), and we conclude by Mukai’s Theorem 4.1.

Lemma 4.7. Let \( \mathcal{F} \) be a GV-sheaf on \( A \). If \( Z \subseteq V^0(\mathcal{F}) \) is an irreducible component whose codimension in \( \text{Pic}^0(X) \) is \( k \), then \( Z \) is a component of \( V^k(\mathcal{F}) \) as well. In particular, \( \dim \text{Supp} \mathcal{F} \geq k \).

Proof. According to Theorem 3.3, since \( \mathcal{F} \) is GV-sheaf, there exists a coherent sheaf \( \mathcal{G} \) on \( \widehat{A} \) such that
\[
R^i \Phi_p \mathcal{F} \simeq \mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\widehat{A}}).
\]
Using Lemma 4.4, this implies that
\[
\text{Supp} \mathcal{G} = \text{Supp} R^i \Phi_p \mathcal{F} = V^0(\mathcal{F}).
\]
Since \( Z \) has codimension \( k \), in combination with Lemma 5.1 we deduce that
\[
R^i \Phi_p \mathcal{F} = 0 \quad \text{for all} \quad i < k
\]
in a neighborhood of \( Z \). But by the GV-condition, we also know that
\[
\text{codim} \text{Supp} R^i \Phi_p \mathcal{F} > k \quad \text{for all} \quad i > k.
\]
It follows that \( Z \) is a component of \( \text{Supp} \ R^k \mathcal{F}_P \), and one can easily deduce by base change that it must be a component of \( V^k(\mathcal{F}) \) as well. \( \square \)

5. Commutative algebra background. In this section I will collect some of the general commutative algebra results used in the lectures. Here \( X \) will always be a smooth variety over an algebraically closed field.\(^2\)

**Lemma 5.1.** If \( \mathcal{F} \) a coherent sheaf on \( X \), then

\[
\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0 \text{ for all } i < \text{codim Supp } \mathcal{F}
\]

and

\[
\text{codim Supp } \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) \geq i \text{ for all } i.
\]

**Proof.** This is a well-known application of the Auslander-Buchsbaum-Serre formula, cf. e.g. [HL97] Proposition 1.1.6(1). The proof of the first assertion is given in loc. cit. in the projective case. For a proof in the local case cf. e.g. [BS76] Proposition 1.17. \( \square \)

**Definition 5.2.** A coherent sheaf \( \mathcal{F} \) on \( X \) is called a \( k \)-th syzygy sheaf if locally there exists an exact sequence

\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{E}_k \rightarrow \ldots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{G} \rightarrow 0
\]

with \( \mathcal{E}_j \) locally free for all \( j \). It is well known for example that 1-st syzygy sheaf is equivalent to torsion-free, and 2-nd syzygy sheaf is equivalent to reflexive. Every coherent sheaf is declared to be a 0-th syzygy sheaf, while a locally free sheaf is declared to be an \( \infty \)-syzygy sheaf.

We consider a variant of Serre’s condition \( S_k \).

**Definition 5.4.** A coherent sheaf \( \mathcal{F} \) on \( X \) satisfies property \( S'_k \) if for all \( x \) in the support of \( \mathcal{F} \) we have:

\[
\text{depth } \mathcal{F}_x \geq \min\{k, \dim \mathcal{O}_{X,x}\}.
\]

The following is a combination of various standard commutative algebra results plus a most likely well-known fact, Lemma 5.7, which is not easily located in the literature. For more general statements, see the Appendix to [Pop12a].

**Proposition 5.5.** For a coherent sheaf \( \mathcal{F} \) on \( X \), the following are equivalent:

(a) \( \mathcal{F} \) is a \( k \)-th syzygy sheaf.

(b) \( \text{codim Supp } \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) \geq i + k \), for all \( i > 0 \).

(c) \( \mathcal{F} \) satisfies \( S'_k \).

**Proof.** For \( k = 0 \), (a) and (c) do not impose any conditions on a coherent sheaf, while (b) also holds since, in any case

\[
\text{codim Supp } \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) \geq i
\]

for any coherent sheaf \( \mathcal{F} \), by Lemma 5.1. The equivalence of (a) and (c) is a basic result of Auslander-Bridger, [AB69] Theorem 4.25. The equivalence of (b) and (c)

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\(^2\)Similar results hold if \( X \) is only assumed to be Cohen-Macaulay, if one replaces the structure sheaf \( \mathcal{O}_X \) by the dualizing sheaf \( \omega_X \).
is [HL97] Proposition 1.1.6(ii) in the case when the support of \( F \) is the entire \( X \). Now for \( k \geq 1 \), conditions (a) and (c) clearly imply that this is the case. We are only left with checking that (b) also implies for \( k \geq 1 \) that \( F \) is supported everywhere. But this follows from the stronger Lemma below. \( \square \)

**Lemma 5.7.** A coherent sheaf \( F \) on \( X \) is torsion-free if and only if

\[
\text{codim Supp } \mathcal{E}xt^i(F, \mathcal{O}_X) > i \quad \text{for all } i > 0.
\]

**Proof.** If \( F \) is torsion free then it is a subsheaf of a locally free sheaf \( E \). From the exact sequence \( 0 \to F \to E \to E/F \to 0 \) it follows that, for \( i > 0 \), \( \mathcal{E}xt^i(F, \mathcal{O}_X) \cong \mathcal{E}xt^{i+1}(E/F, \mathcal{O}_X) \). But then (5.6), applied to \( E/F \), implies that

\[
\text{codim Supp } \mathcal{E}xt^i(F, \mathcal{O}_X) > i, \quad \text{for all } i > 0.
\]

Conversely, since \( X \) is smooth, the functor \( R\mathcal{H}om(\cdot, \mathcal{O}_X) \) is an involution. Thus there is a spectral sequence

\[
E_2^{ij} := \mathcal{E}xt^i \left( \mathcal{E}xt^j(F, \mathcal{O}_X), \mathcal{O}_X \right) \Rightarrow H^{i-j} F = \begin{cases} F & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
\]

If \( \text{codim Supp } \mathcal{E}xt^i(F, \mathcal{O}_X) > i \) for all \( i > 0 \), then \( \mathcal{E}xt^i \left( \mathcal{E}xt^j(F, \mathcal{O}_X), \mathcal{O}_X \right) = 0 \) for all \( i, j \) such that \( j > 0 \) and \( i - j \leq 0 \), so the only \( E_\infty^{00} \) term which might be non-zero is \( E_\infty^{00} \). But the differentials coming into \( E_\infty^{00} \) are always zero, so we get a sequence of inclusions

\[
\mathcal{F} = H^0 = E_\infty^{00} \subset \ldots \subset E_3^{00} \subset E_2^{00}.
\]

The extremes give precisely the injectivity of the natural map \( \mathcal{F} \to \mathcal{F}^{**} \). Hence \( \mathcal{F} \) is torsion free. \( \square \)

In the proof of Theorem 7.4 we used the sheaf theoretic version of the Syzygy Theorem of Evans-Griffith.

**Theorem 5.8** ([EG81], Corollary 1.7). Let \( \mathcal{F} \) be a \( k \)-th syzygy sheaf on \( X \) which is not locally free. Then \( \text{rank}(\mathcal{F}) \geq k \).

**B. Application: a birational characterization of abelian varieties**

**6. Characterization of abelian varieties.**

**Definition 6.1.** The \( m \)-th plurigenus of \( X \) is \( P_m(X) = h^0(X, \omega_X^\otimes m) \).

**Theorem 6.2** (Chen-Hacon, [CH01]). If \( X \) is a smooth projective complex variety with \( q(X) = \dim X \) and \( P_1(X) = P_2(X) = 1 \), then \( X \) is birational to an abelian variety.

**Remark.** Weaker results had been obtained previously: for instance, a result of Kawamata stated that \( X \) is birational to an abelian variety if \( q(X) = \dim X \) and \( \kappa(X) = 0 \). This was improved by Kollár, who replaced the hypothesis \( \kappa(X) = 0 \) with \( P_m(X) = 1 \) for some \( m \geq 3 \); he conjectured the even stronger result proved by Chen and Hacon.
I will present a proof of Theorem 6.2 due to G. Pareschi, in which the result is a consequence of the following collection of statements. We always assume in this section that $X$ is a smooth projective complex variety.

**Proposition 6.3** (Ein-Lazarsfeld). If $P_1(X) = P_2(X) = 1$, then $0$ is an isolated point in $V^0(\omega_X)$.

**Proof.** First note that $0 \in V^0(\omega_X)$, since $P_1(X) \neq 0$. Assume now that there exists a positive dimensional component $Z \subset V^0(\omega_X)$ with $0 \in Z$. By the Linearity Theorem 2.3, $Z$ is an abelian subvariety of Pic$^0(X)$. In particular, $L \in Z \implies L^{-1} \in Z$.

Now for any $L \in Z$ consider the multiplication map

$$H^0(X, \omega_X \otimes L) \otimes H^0(X, \omega_X \otimes L^{-1}) \to H^0(X, \omega_X^{\otimes 2}).$$

By assumption $L$ moves in a positive dimensional family, while the vector space on the right hand side is isomorphic to $\mathbb{C}$, since $P_2(\omega_X) = 1$. This is a contradiction: indeed, if $D$ is the unique divisor in $|2K_X|$, sections in the image of the multiplication map would produce an infinitely many components of $D$ as $L$ varies. \hfill \Box

**Proposition 6.4** (Ein-Lazarsfeld). If $0$ is isolated in $V^0(\omega_X)$, then the Albanese map of $X$ is surjective.

**Proof.** By the projection formula one has $V^0(X, \omega_X) = V^0(A, a_\ast \omega_X)$, so $0$ is isolated in $V^0(a_\ast \omega_X)$. On the other hand, $a_\ast \omega_X$ satisfies the Dimension Theorem, according to Hacon’s result. By Lemma 4.4, denoting $g = \dim \text{Alb}(X)$, we have

$$V^g(a_\ast \omega_X) \subseteq V^{g-1}(a_\ast \omega_X) \subseteq \cdots \subseteq V^0(a_\ast \omega_X).$$

Since $\{0\}$ has codimension $g$ in Pic$^0(X)$, Lemma 4.7 implies that $0 \in V^g(a_\ast \omega_X)$ as well, which then gives $\dim a(X) \geq g$. This can happen only when $a(X) = \text{Alb}(X)$. \hfill \Box

**Proposition 6.5** (Pareschi). If $X$ is of maximal Albanese dimension and $V^0(\omega_X)$ is 0-dimensional, then the Albanese map of $X$ is birational.

**Proof.** Denote $n = \dim X$ and $g = \dim A$. Combining Proposition 6.4 with the fact that $X$ is of maximal Albanese dimension, we get that $n = g$. It is enough to show that

$$a_\ast \omega_X \simeq \mathcal{O}_A,$$

since this implies that $a$ has degree 1. Since $a$ is generically finite, the Grauert-Riemenschneider Theorem 3.1 guarantees that

$$R^i a_\ast \omega_X = 0 \quad \text{for all } i > 0.$$

This implies that the Leray spectral sequence degenerates, so that for all $L \in \text{Pic}^0(A)$:

$$H^i(A, a_\ast \omega_X \otimes L) \simeq H^i(X, \omega_X \otimes a^\ast L) \quad \text{for all } i,$$

and hence $V^i(\omega_X) = V^i(a_\ast \omega_X)$ for all $i$. Now all the points in $V^0(a_\ast \omega_X)$ are isolated by hypothesis, and so by Lemma 4.7 they all belong to $V^g(a_\ast \omega_X) = V^g(\omega_X) = V^n(\omega_X) = \{0\}$. \hfill \Box
Finally this implies that
\[ V^i(\omega_X) = \{0\} \text{ for all } i. \]
Going back to the criterion in Theorem 3.3, it follows that
\[ R^i\Phi_P(a_*\omega_X) = 0 \text{ for all } i < g. \]
On the other hand, by the base change theorem \( R^g\Phi_P(a_*\omega_X) \) is supported at \( \{0\} \), and in fact \( R^g\Phi_P(a_*\omega_X) \simeq \mathcal{O}_{\{0\}} \); see the Exercise below. It follows by Exercise 6.6 that we have an isomorphism
\[ R\Phi_P(a_*\omega_X) \simeq R\Phi_P(\mathcal{O}_A), \]
and that this implies \( a_*\omega_X \simeq \mathcal{O}_A \). \( \square \)

**Exercise 4.** Let \( X \) be a smooth projective variety of dimension \( n \). Then
\[ R^n\Phi_P\omega_X \simeq \mathcal{O}_{\{0\}}. \]

In order to conclude the proof of Theorem 6.2, due to the previous results we are left with showing that if \( P_1(X) = P_2(X) = 1 \) and \( q(X) = \dim X \), then \( V^0(\omega_X) \) is zero dimensional. This follows by the exact same argument used in Proposition 6.3, once we show the following:

**Proposition 6.7.** Assume that \( X \) is of maximal Albanese dimension and \( V^0(\omega_X) \) is positive dimensional. Then there exists a subvariety \( Z \subset \text{Pic}^0(X) \) with \( \dim Z > 0 \) such that \( Z \) and \( -Z \) are both contained in \( V^0(\omega_X) \).

**Proof.** If \( V^0(\omega_X) \) has a positive dimensional component \( Z' \) passing through the origin, then by the Linearity Theorem it is an abelian subvariety, and therefore we can take \( Z = Z' \).

Assume now that none of the positive dimensional components of \( V^0(\omega_X) \) contains the origin, and take \( W \) to be one of them, say of codimension \( k \). Again by the Linearity Theorem,
\[ W = \lfloor L \rfloor + B, \]
where \( B \subset \text{Pic}^0(X) \) is an abelian subvariety and \( L \) is a torsion line bundle. Take \( C = \text{Pic}^0(B) \); it is an abelian variety of dimension \( q(X) - k \). If \( p : A = \text{Alb}(X) \to C \) is the natural projection, we can consider the composition
\[ X \xrightarrow{a} A \xrightarrow{f} C \xrightarrow{p} \]
By construction we have \( \dim f(X) = d - k \), where \( d = \dim X \). For line bundles \( \alpha \in \text{Pic}^0(C) \), the degeneration of the Leray spectral sequence given by the Remark following Theorem 1.4 implies
\[ H^k(X, \omega_X \otimes L \otimes f^*\alpha) \simeq \bigoplus_{j=0}^k H^{k-j}(C, R^j f_*(\omega_X \otimes L) \otimes \alpha). \]
On the other hand, \( W \subset \text{Pic}^0(X) \) consists precisely of line bundles of the form \( L \otimes f^* \alpha \) as above, and therefore the left hand side is nonzero for all \( \alpha \), consequently
\[
\text{Pic}^0(C) = \bigcup_{j=0}^{k} V^{k-j}(C, R^j f_*(\omega_X \otimes L)).
\]

But according to Hacon’s theorem, \( R^j f_*(\omega_X \otimes L) \) is a \( GV \)-sheaf on \( \text{Pic}^0(C) \), and so for dimension reasons we must have
\[
\text{Pic}^0(C) = V^0(C, R^k f_*(\omega_X \otimes L)).
\]

In particular, certainly \( R^k f_*(\omega_X \otimes L) \neq 0 \).

Now the claim is that \( R^k f_*(\omega_X \otimes L^{-1}) \neq 0 \) as well. To this end, consider the Stein factorization of \( f \):
\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
& \searrow & \downarrow h \\
& & C
\end{array}
\]

and denote by \( F \) the general fiber of \( g \); it is connected and \( k \)-dimensional. Since \( R^k f_*(\omega_X \otimes L) \neq 0 \), we have by base change that \( H^k(F, \omega_F \otimes L|_F) \neq 0 \), which by Serre duality implies \( L|_F \simeq \mathcal{O}_F \). This means \( L^{-1}|_F \simeq \mathcal{O}_F \) as well, and reversing the argument proves the claim.

The next thing we will show is that
\[
(6.8) \quad \dim V^0(C, R^k f_*(\omega_X \otimes L^{-1})) > 0.
\]
First, since \( R^k f_*(\omega_X \otimes L^{-1}) \) is \( GV \) and non-zero, it follows from Lemma 4.6 that \( V^0(C, R^k f_*(\omega_X \otimes L^{-1})) \neq \emptyset \). But if \( \alpha \in V^0(C, R^k f_*(\omega_X \otimes L^{-1})) \) is an isolated point, then according to Lemma 4.7 \( \alpha \) also belongs to \( V^{g-k}(C, R^k f_*(\omega_X \otimes L^{-1})) \). Since \( \dim f(X) = d - k \), we get that \( d = g \) and \( f(X) = C \). Moreover
\[
0 \neq H^{n-k}(C, R^k f_*(\omega_X \otimes L^{-1}) \otimes \alpha) \subseteq H^n(X, \omega_X \otimes L^{-1} \otimes f^* \alpha),
\]
where the second inclusion follows as above from Kollár’s theorem. We deduce that
\[
L \simeq f^* \alpha \in f^* \text{Pic}^0(C),
\]
which is a contradiction with the initial choice of \( L \).

We are now ready to conclude the proof: pick a positive dimensional component \( T \) of \( V^0(C, R^k f_*(\omega_X \otimes L^{-1})) \), and define
\[
Z := f^* T \subset V^k(\omega_X).
\]
In other words
\[
Z \subseteq [L^{-1}] + f^* \text{Pic}^0(C) \subset V^0(\omega_X),
\]
but also
\[
-Z \subseteq [L] + f^* \text{Pic}^0(C) = W \subset V^0(\omega_X).
\]
\( \square \)
C. Application: a higher dimensional Castelnuovo-de Franchis theorem

7. Higher dimensional Castelnuovo-de Franchis. The search for bounds for the Hodge numbers of an irregular variety in terms of the number of independent holomorphic 1-forms has a long history, going back at least as far as the classical inequality of Castelnuovo and de Franchis giving a lower bound on the geometric genus of surfaces \( S \) without irrational pencils, namely

\[
p_g(S) = h^{0,2}(S) \geq 2q(S) - 3.
\]

This was extended by several authors; for instance, Catanese [Cat91] showed that for a compact Kähler manifold with no irregular fibrations, the natural maps

\[
\phi_k : \bigwedge^k H^1(X, \mathcal{O}_X) \longrightarrow H^k(X, \mathcal{O}_X)
\]

are injective on primitive forms \( \omega_1 \wedge \ldots \wedge \omega_k \), for \( k \leq \dim X \). Since these correspond to the Plücker embedding of the Grassmannian \( G(k, V) \), one obtains the bounds

\[
h^{0,k}(X) \geq k(q(X) - k) + 1.
\]

In dimension at least three, one can consider an generalization of the Castelnuovo-de Franchis inequality going in a different direction. Note first that the statement for surfaces can be rewritten as

\[
\chi(\omega_S) \geq q(S) - 2.
\]

In this form, we can extend it to the following:

**Theorem 7.2** (Pareschi – P., [PP09]). Let \( X \) be a smooth projective variety without irregular fibrations. Then

\[
\chi(\omega_X) \geq q(X) - \dim X.
\]

**Definition 7.3.** Let \( X \) be a smooth projective variety of maximal Albanese dimension. The **generic vanishing index** of \( \omega_X \) is

\[
gv(\omega_X) := \min_{i > 0} \{ \text{codim } V^i(\omega_X) - i \}.
\]

We also introduce a version taking into account only the union \( V^i(\omega_X)_0 \) of the irreducible components containing the origin:

\[
gv_0(\omega_X) := \min_{i > 0} \{ \text{codim } V^i(\omega_X)_0 - i \}.
\]

Note that by the Dimension Theorem

\[
gv_0(\omega_X) \geq gv(\omega_X) \geq 0.
\]

To prove Theorem 7.2, note first that the Fibrations Theorem 2.4 implies that if \( X \) has no irregular fibrations, then \( 0 \) is an isolated point in \( V^i(\omega_X) \) for all \( i > 0 \). This immediately gives in this case that

\[
gv_0(\omega_X) = q(X) - \dim X.
\]

Therefore Theorem 7.2 is a consequence of the following more general

**Theorem 7.4.** If \( X \) is of maximal Albanese dimension, then

\[
\chi(\omega_X) \geq gv_0(\omega_X).
\]
Proof. Recall from the Remark after Theorem 3.5 that if $X$ is of maximal Albanese dimension, we have that $\mathcal{R}\Phi_P \mathcal{O}_X$ is a sheaf $\widehat{\mathcal{O}}_X$, supported in degree $d = \dim X$. In other words

$$\widehat{\mathcal{O}}_X \simeq \mathcal{R}^d p_{\ast} P,$$

where $P$ is a poincaré bundle on $X \times \hat{A}$. By cohomology and base change we have that for a general $L \in \text{Pic}^0(X)$, the fiber of $\widehat{\mathcal{O}}_X$ at $[L]$ is

$$H^d(X, L) \simeq H^0(X, \omega_X \otimes L^{-1})^\vee.$$

As in previous arguments, it follows then from the Dimension Theorem that the generic rank of $\widehat{\mathcal{O}}_X$ is

$$\text{rk } \widehat{\mathcal{O}}_X = \chi(\omega_X).$$

Now by Lemma 4.5 and the definition of the generic vanishing index, around the origin

$$\text{codim } \text{Supp}_0 \mathcal{R}^i \Phi_P \omega_X \geq i + \text{gv}_0(\omega_X).$$

But we also know from (3.8) that

$$\mathcal{R}^i \Phi_P \omega_X \simeq \text{Ext}^i(\widehat{\mathcal{O}}_X, \mathcal{O}_{\hat{A}}).$$

Proposition 5.5 implies in turn that $\widehat{\mathcal{O}}_X$ is a $\text{gv}_0(\omega_X)$-syzygy sheaf in a neighborhood of the origin. Let’s also convince ourselves that $\widehat{\mathcal{O}}_X$ is not locally free at the origin; indeed, $\widehat{\mathcal{O}}_X$ being locally free would be equivalent to the case $k = \infty$ in Proposition 5.5, which would imply that $\mathcal{R}^i \Phi_P \omega_X = 0$ for all $i > 0$ around the origin. But in fact $\mathcal{R}^d \Phi_P \omega_X \simeq \mathcal{O}_{\{0\}}$, according to Exercise 4.

The result follows now from the Syzygy Theorem 5.8 applied to $\widehat{\mathcal{O}}_X$. □

References


NOTES FOR THE ISTANBUL LECTURES


