DEFORMATIONS OF POLARIZED AUTOMORPHIC GALOIS REPRESENTATIONS
AND ADJOINT SELMER GROUPS

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Abstract. We prove the vanishing of the geometric Bloch–Kato Selmer group for the adjoint representation
of a Galois representation associated to regular algebraic polarized cuspidal automorphic representations
under an assumption on the residual image. Using this, we deduce that the localization and completion of
a certain universal deformation ring for the residual representation at the characteristic zero point induced
from the automorphic representation is formally smooth of the correct dimension. We do this by employing
the Taylor–Wiles–Kisin patching method together with Kisin’s technique of analyzing the generic fibre of
universal deformation rings. Along the way we give a characterization of smooth closed points on the generic
fibre of Kisin’s potentially semistable local deformation rings in terms of their Weil–Deligne representations.

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Introduction

Let $F$ be a number field, $S$ a finite set of finite places of $F$ containing all those above a fixed rational
prime $p$, and let $F(S)$ be the maximal extension of $F$ unramified outside of $S$ and the Archimedean places.
Given a $p$-adic representation $V$ of $\text{Gal}(F(S)/F)$, Bloch and Kato [BK] defined certain subspaces

$$H_{f}^{1}(F(S)/F, V) \subseteq H_{g}^{1}(F(S)/F, V) \subseteq H^{1}(F(S)/F, V)$$

of the Galois cohomology group $H^{1}(F(S)/F, V)$, known as the Bloch–Kato Selmer group and geometric
Bloch–Kato Selmer group, respectively. If $V$ is deRham, resp. crystalline, then $H_{f}^{1}(F(S)/F, V)$, resp.
$H_{f}^{1}(F(S)/F, V)$, is the subspace of $H^{1}(F(S)/F, V) = \text{Ext}_{\mathbb{Q}_{p}[\text{Gal}(F(S)/F)]}^{1}(\mathbb{Q}_{p}, V)$ of extensions of the trivial
representation by $V$ that are deRham, resp. crystalline. When $V$ is deRham (so conjecturally also pure),
Bloch and Kato made a far reaching and influential conjecture that relates the dimension of $H_{f}^{1}(F(S)/F, V)$
to the order of vanishing of the $L$-function of the dual representation of $V$ at the point $s = 1$. One prediction
of this conjecture is that if the representation $V$ is pure of motivic weight zero, then

$$H_{f}^{1}(F(S)/F, V) = H_{g}^{1}(F(S)/F, V) = 0.$$
This is in accordance with a philosophy of Grothendieck that in a conjectural category of mixed motives, there should be no nontrivial extensions of pure motives of the same weight.

Let $E$ be a finite extension of $\mathbb{Q}_p$. Given any continuous pure deRham representation

$$\rho : \text{Gal}(F(S)/F) \to \text{GL}_d(E),$$

one naturally obtains a pure weight zero representation called the adjoint representation: $\text{ad}(\rho) = \text{gl}_d(E)$, the Lie algebra of $\text{GL}_d(E)$, with $\text{Gal}(F(S)/F)$-action given by composing $\rho$ with the adjoint action of $\text{GL}_d(E)$ on its Lie algebra. Then the Bloch–Kato conjecture predicts

$$H^1_c(F(S)/F, \text{ad}(\rho)) = 0.$$

In the case where $\rho$ is the representation arising from an elliptic curve over $\mathbb{Q}$, this prediction was first proved by Flach [Fla92] by a method using Euler systems, assuming that the elliptic curve has good reduction at $p$, that $p \geq 5$, and that the associated residual representation

$$\overline{\rho} : \text{Gal}(\mathbb{Q}(S)/\mathbb{Q}) \to \text{GL}_d(\mathbb{F}_p)$$

is surjective. A corollary of the breakthrough work of Wiles and Taylor–Wiles is this vanishing in the case that $\rho$ is the representation coming from a modular form of weight $k \geq 2$ and level $\Gamma_1(N)$ with $p > 2$, $p^2 \nmid N$, and such that the residual representation satisfies the Taylor–Wiles hypothesis: that the restriction of $\overline{\rho}$ to $\text{Gal}(F(S)/F(\zeta_p))$ is absolutely irreducible, where $\zeta_p$ denotes a primitive $p$th root of unity. This results from their so-called $R = \mathbb{T}$ theorem that equates a certain universal deformation ring of $\overline{\rho}$ to a Hecke algebra. On way to think about this is that the tangent space of the deformation ring they consider at the characteristic zero point corresponding to the modular form is equal to its adjoint Bloch–Kato Selmer group, while the tangent space of the Hecke algebra at that point is trivial, since the Hecke algebra is reduced. After some intermittent work, Kisin [Kis04] showed the vanishing of $H^1_c(G_{\mathbb{Q}, S}, \text{ad}(\rho))$ for modular forms of weight $k \geq 2$ and arbitrary level, assuming only a mild condition on the residual representation. His method uses some of the ideas of Taylor–Wiles coupled with a careful analysis of the integral étale cohomology of modular curves. We mention also the result of Weston [Wes04] which applies to non-CM forms with certain hypotheses on the level, but has no restriction on the residual representation. For general totally real, resp. CM fields, but still in dimension two, one can deduce results of this form from the $R[1/p] = \mathbb{T}[1/p]$ theorems [Kis09a, Theorem 3.4.11] and [KW09, Propositions 9.2 and 9.3], resp. [GK14, Corollary 3.4.3], whenever the assumptions of those theorems are satisfied.

In higher dimensions, one is naturally led to consider the Galois representations associated to regular algebraic polarized cuspidal automorphic representations of $\text{GL}_d$ over CM fields. These adjoint representations have a natural extension to a representation of the Galois group of the maximal totally real subfield, and this adjoint Selmer group has a natural interpretation as the tangent space of a polarized deformation ring. Although there has been great progress in modularity lifting in this context, almost all of the results prove only $R_{\text{red}}^{\text{cusp}} = \mathbb{T}$ and do not imply vanishing of the adjoint Selmer groups, although some cases can be deduced using the $R = \mathbb{T}$ theorem of Clozel–Harris–Taylor [CHT08, Theorem 3.5.1]. Using a completely different method, namely the theory of eigenvarieties, Chenier [Che11, Theorem F] proved that the vanishing of this adjoint Selmer group is equivalent to the vanishing of the $H^2$ of this adjoint representation under some local hypotheses. In particular, when the deformation problem is unobstructed (for example, see [Che11, Appendix]) one deduces the vanishing of this adjoint Bloch–Kato Selmer group under these hypotheses.

The main observation used in this paper is that one can still use the Taylor–Wiles–Kisin patching method to deduce vanishing of the corresponding adjoint Selmer groups for automorphic Galois representations provided one knows that induced points on local deformation rings are smooth. Indeed, the method yields a ring $R_{\infty}$, an $R_{\infty}$-module $M_{\infty}$, and a control theorem that relates them to our deformation ring and a space of automorphic forms. The most subtle part in proving modularity lifting theorems is to understand the components of $R_{\infty}$ and how they relate to congruences between automorphic forms. But if we are only interested in the infinitesimal deformation theory of the characteristic zero point coming from our automorphic Galois representation $\rho$, we can localize and complete at this point, and if we know that $\rho$ determines a smooth point on the local deformation rings, it also determines a smooth point on $R_{\infty}$. Then we can apply the Auslander–Buchsbaum formula to the completion and deduce that the localized and completed deformation ring acts freely on a finite dimensional vector space of cusp forms, from which we can deduce vanishing of the adjoint Selmer group. Before stating our main theorems, we set up some notation.
Let $E$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. Let $F$ be a CM field with maximal totally real subfield $F^+$. Let $\overline{F}$ be some fixed algebraic closure of $F$, and let $c \in \text{Gal}(\overline{F}/F^+)$ be a choice of complex conjugation. Let $S$ be a finite set of finite places of $F^+$ containing all places above $p$. Let $F(S)$ be the maximal extension of $F$ unramified outside of the places in $F$ above those in $S$. Note that $F(S)$ is Galois over $F^+$. Let

$$\rho : \text{Gal}(F(S)/F) \rightarrow \text{GL}_d(\mathbb{E})$$

be a continuous, absolutely irreducible representation, and let $\text{ad}(\rho)$ denote the Lie algebra $\mathfrak{gl}_d(\mathbb{E})$ of $\text{GL}_d(\mathbb{E})$ with the adjoint action $\text{ad} \circ \rho$ of $\text{Gal}(F(S)/F)$.

We assume there is a continuous totally odd character $\mu : G_{F^+} \rightarrow E^\times$ and an invertible symmetric matrix $P$ such that the pairing $\langle a, b \rangle = \langle aP^{-1}b \rangle_{E^d}$ is perfect and satisfies

$$(\rho(\sigma)a, \rho(c\sigma)b) = \mu(\sigma)(a, b),$$

for all $\sigma \in \text{Gal}(F(S)/F)$. Since $\rho$ is absolutely irreducible, $P$ is unique up to scalar. We can then extend the action of $\text{Gal}(F(S)/F)$ on $\text{ad}(\rho)$ to an action of $\text{Gal}(F(S)/F^+)$ by letting $c$ act by $X \mapsto -P^T XP^{-1}$, and this is independent of the choice of $c$ and of $P$.

Finally, we recall that we can choose a $\text{Gal}(F(S)/F)$-stable $\mathcal{O}$-lattice in the representation space of $\rho$, so after conjugation we may assume that $\rho$ takes values in $\text{GL}_d(\mathcal{O})$. The semisimplification of its reduction modulo the maximal ideal of $\mathcal{O}$ does not depend on the choice of lattice, and we denote it by $\overline{\rho} : \text{Gal}(F(S)/F) \rightarrow \text{GL}_d(\mathbb{F})$.

**Theorem A.** Assume $p > 2$. Assume there is a finite extension $L/F$ of CM fields, a regular algebraic polarizable cuspidal automorphic representation $\Pi$ of $\text{GL}_d(\mathcal{A}_L)$, and an isomorphism $\iota : \overline{\rho} \cong \mathbb{C}$ such that the following hold:

1. $\rho|_{G_L} \cong \overline{\rho}_{1,\iota}$, where $\rho_{1,\iota}$ is the Galois representation attached to $\Pi$ and $\iota$;
2. $\xi_p \not\in L$ and $\overline{\rho}(G_{L_{Q_p}})$ is adequate.

Then

1. the geometric Bloch–Kato Selmer group

$$H^1_g(F(S)/F^+, \text{ad}(\rho)) := \ker(H^1(F(S)/F^+, \text{ad}(\rho)) \rightarrow \prod_{v|p} H^1(F^+_v, B_{d\mathbb{R}} \otimes \mathbb{Q}_p, \text{ad}(\rho)))$$

is trivial.

Moreover, \(H^2(F(S)/F^+, \text{ad}(\rho) = 0;\)

2. letting $H^1_g(F^+_v, \text{ad}(\rho)) := \ker(H^1(F^+_v, \text{ad}(\rho)) \rightarrow H^1(F^+_v, B_{d\mathbb{R}} \otimes \mathbb{Q}_p, \text{ad}(\rho)))$ for each $v|p$ in $F^+$, the natural map

$$H^1(F(S)/F^+, \text{ad}(\rho)) \rightarrow \prod_{v|p} H^1(F^+_v, \text{ad}(\rho))/H^1_g(F^+_v, \text{ad}(\rho))$$

is an isomorphism.

We refer the reader to the notation section below for any of the notation with which the reader is not familiar, to 2.1 for a discussion of regular algebraic polarizable cuspidal automorphic representations and their associated Galois representations, and to 3.1.1 for the definition of an adequate subgroup of $\text{GL}_d(\mathbb{F})$. If $p > 2(d + 1)$, then any subgroup of $\text{GL}_d(\mathbb{F})$ acting absolutely irreducibly on $\mathbb{F}^d$ is adequate by a theorem of Guralnick–Herzig–Taylor–Thorne [Tho12, Theorem A.9], and the assumption of potential automorphy is satisfied in many cases by work of Barnet–Lamb–Gee–Geraghty–Taylor [BLGGT14, Theorem 4.5.1]. Parts 2. and 3. follow from part 1. using an argument of Kisin (see 1.3.5), and this relies on the “numerical coincidence” discussed in [CHT08, §1]. Theorem A will be deduced in §3.1 (see 3.2.1) from a slight variant (3.1.5).

For the $\rho$ as in the statement of the theorem, $H^1_g(F^+_v, \text{ad}(\rho)) = H^1_{f}(F^+_v, \text{ad}(\rho))$ for any $v|p$ in $F^+$, where

$$H^1_{f}(F^+_v, \text{ad}(\rho)) := \ker(H^1(F^+_v, \text{ad}(\rho)) \rightarrow H^1(F^+_v, B_{cr} \otimes \mathbb{Q}_p, \text{ad}(\rho))),$$

is the more common local Bloch–Kato Selmer group (see 1.2.9), so Theorem A remains unchanged replacing $H^1_g$ with $H^1_f$ everywhere. With this in mind, the reader should compare Theorem A with the results of
Moreover, such that: the adjoint Gal(\mathbb{Q}_p \otimes \overline{\mathbb{Q}}_p \cong \rho_{\pi,t}, \text{ where } \rho_{\pi,t} \text{ is the Galois representation attached to } \pi \text{ and } t; 
\begin{enumerate}
\item the geometric Bloch–Kato Selmer group
\[ H^1_{\text{fl}}(F^+(S)/F^+, \text{ad}((\rho))) := \ker(H^1(F^+(S)/F^+, \text{ad}(\rho)) \to \prod_{v|p} H^1(F^+_v, B_{\text{dR}} \otimes \mathbb{Q}_p, \text{ad}(\rho))) \]
\] is trivial. Moreover,
\item \[ H^2(F^+(S)/F^+, \text{ad}^0(\rho)) = 0; \]
\item letting \[ H^1_{\text{fl}}(F^+_v, \text{ad}^0(\rho)) := \ker(H_1(F^+_v, \text{ad}^0(\rho)) \to H^1(F^+_v, B_{\text{dR}} \otimes \mathbb{Q}_p, \text{ad}^0(\rho))) \] for each \( v \mid p \) in \( F^+ \), the natural map \[ H^1(F^+(S)/F^+, \text{ad}^0(\rho)) \to \prod_{v|p} H^1(F^+_v, \text{ad}^0(\rho))/H^1_{\partial}(F^+_v, \text{ad}^0(\rho)) \]
is an isomorphism.
\end{enumerate}

As in [Kis04, §8], we use vanishing of the geometric Bloch–Kato adjoint Selmer group to deduce smoothness of a certain universal deformation ring at automorphic points, something we now explain. Let \( \mathcal{G}_d \) denote the group scheme over \( \mathbb{Z} \) that is the semidirect product

\[ \mathcal{G}^0_\partial \rtimes \{1, j\} = (\text{GL}_d \times \text{GL}_1) \rtimes \{1, j\}, \]

where \( j(g, a) = (a, g^{-1}, a) \). There is a character \( \nu : \mathcal{G}_d \to \text{GL}_1 \) defined by by \( \nu(g, a) = a \) and \( \nu(j) = -1 \). Fix a continuous homomorphism
\[ \overline{\tau} : \text{Gal}(F(F^+)/F^+) \to \mathcal{G}_d(F) \]
inducing an isomorphism \( \text{Gal}(F(F^+)/F^+) \cong \mathcal{G}_d(F)/\mathcal{G}^0_\partial(F) \), and a continuous totally odd deRham character \( \mu : \text{Gal}(F(F^+)/F^+) \to \mathcal{O}^\times \) such that \( \nu \circ \overline{\tau} = \mu \mod \mathcal{O} \). Assume that the composite of \( \overline{\tau}_{G_F} : \text{Gal}(F(F)/F) \to \mathcal{G}^0_\partial(F) \) with the projection \( \mathcal{G}^0_\partial(F) \to \text{GL}_d(F) \) is absolutely irreducible. If \( p > 2 \), then with this data \( S = (F/F^+, S, \mathcal{O}, \overline{\tau}, \mu) \), there is a complete Noetherian local commutative \( \mathcal{O} \)-algebra \( R_S \) with residue field \( F \), such that \( R_S \) represents the set-valued functor on the category of complete Noetherian local commutative \( \mathcal{O} \)-algebras \( A \) with residue field \( F \), that sends \( A \) to the set deformations of \( \overline{\tau} \) to \( A \) with multiplier \( \mu \), i.e. the set of \( 1 + \mathcal{M}_d(m_A) \)-conjugacy classes of homomorphisms
\[ r : \text{Gal}(F(F)/F^+) \to \mathcal{G}_d(A) \]
Theorem C. Assume \( p > 2 \). Let \( x \) be a closed point of \( \text{Spec} \, R_{\mathbb{Q}}[1/p] \) with residue field \( k \), and let

\[
r_x : \text{Gal}(F(S)/F^+) \rightarrow \mathcal{G}_d(k)
\]

be the pushforward of (some homomorphism representing) the universal deformation of \( \mathcal{F} \) via \( R_{\mathbb{Q}}[1/p] \rightarrow k \). Let \( \rho \) denote the composite of \( r|_{\mathcal{G}_L} : \text{Gal}(F(S)/F) \rightarrow \mathcal{G}_0^0(k) \) with the projection \( \mathcal{G}_0^0(k) \rightarrow \text{GL}_d(k) \).

Assume there is a finite extension \( L/F \) of CM fields, a regular algebraic polarizable cuspidal automorphic representation \( \Pi \) of \( \text{GL}_d(\mathbb{A}_L) \), and an isomorphism \( \iota : \mathbb{G}_p \xrightarrow{\sim} \mathbb{C} \) such that the following hold:

\[
\begin{align*}
(a) \quad & \rho|_{\mathcal{G}_L} \otimes \mathbb{G}_p \cong \rho_{\Pi, \iota}; \\
(b) \quad & \zeta_p \notin L \text{ and } \tau(G_{L(G_p)}) \text{ is adequate.}
\end{align*}
\]

Then the localization and completion \( (R_S)_x \) is formally smooth over \( k \) of dimension \( \frac{d(d+1)}{2} \lfloor F^+ : \mathbb{Q} \rfloor \).

The reader is referred to 3.1.2 for the definition of what it means for a subgroup of \( \mathcal{G}_d(\mathbb{F}) \) to be adequate. Theorem C is deduced in 3.2.3 as an immediate consequence of 3.1.5 and 1.3.13.

A related application of Theorem A is to the geometry of unitary eigenvarieties. Bellaïche and Chenevier showed that at certain classical automorphic points on a unitary eigenvariety (see [BC09, §7.6.2]), the weight map is étale and that a “refined deformation ring” (see [BC09, Definition 7.6.2 and Proposition 7.6.3]) is isomorphic to the completed local ring of the structure sheaf of the eigenvariety at these points [BC09, Corollary 7.6.11], provided that [BC09, Conjecture 7.6.5 (C1)] holds. Theorem A implies [BC09, Conjecture 7.6.3]) is isomorphic to the completed local ring of the structure sheaf of the eigenvariety at these points.
Theorems A, B, and C, we will only need the direction “generic implies smooth”, but the author has decided to include the converse here because he finds it interesting in its own right: the criterion is the same as in the \( \ell \neq p \) case, exhibiting a sort of “independence of \( p \)” phenomena for the geometry of potentially semistable deformation rings.

Outline. The paper is organized as follows.

In §1, we recall and prove the relevant facts regarding the deformation theory of Galois representations. We first recall some properties of Weil–Deligne representations in 1.1; namely the construction of Weil–Deligne representations and their connection with smooth admissible representations of \( GL_d \). In §1.2, we treat the local theory of Galois deformations with an emphasis on describing the smooth points in the generic fibre of local deformation rings; in particular, we prove Theorem D (1.2.7). The global theory is then discussed in §1.3, the main point being to use Kisin’s technique of analyzing the generic fibre of deformation rings to connect them to the Bloch–Kato Selmer group (1.3.12), and to prove some dimension and smoothness results that are necessary for the proof of Theorem C (1.3.13).

We discuss the automorphic theory necessary in §2; everything here is standard. We first recall the properties of Galois representations associated to regular algebraic polarized cuspidal automorphic representations of CM and totally real fields in §2.1. We then describe their connection to automorphic forms on definite unitary groups in §2.2, the main point being to show how one can construct a certain module of automorphic forms with an action of a certain Galois deformation ring, and that the vanishing of the Bloch–Kato Selmer group in question is implied by a freeness property (2.2.7).

We conclude our efforts in §3, proving the main theorem (3.1.5) in §3.1, and deducing Theorems A, B, and C from the introduction in §3.2.

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Notation and conventions. If \( F \) is a number field and \( v \) is a place of \( F \), we denote by \( F_v \) the completion of \( F \) at \( v \), and if \( v \) is non-Archimedean we denote the ring of integers by \( \mathcal{O}_{F_v} \). We let \( \mathcal{A}_F \) denote the ring of adeles of \( F \), and \( \mathcal{A}_F^\infty \) the ring of finite adeles.

If \( K \) is any field with a fixed algebraic closure \( \overline{K} \), we denote by \( G_K \) the absolute Galois group \( \text{Gal}(\overline{K}/K) \).

If \( K \) is a non-Archimedean local field, we denote by \( I_K \) the inertia subgroup, by \( W_K \) the Weil group, and by \( \text{Frob}_v \) the geometric Frobenius in \( G_K/I_K \cong W_K/I_K \). In the case that \( K \) is the completion of a number field \( F \) at a finite place \( v \), we write \( G_v, I_v, \text{Frob}_v \) for \( G_{F_v}, I_{F_v}, \text{Frob}_{F_v} \), respectively. If \( L/F \) is a Galois extension of number fields inside some fixed algebraic closure \( \overline{F} \) of \( F \), and \( S \) is a finite set of finite places of \( F \), we let \( L(S) \) denote the maximal extension of \( L \) that is unramified outside of any of the places in \( S \) above those in \( S \) and the Archimedean places. A CM extension of a totally real field is always assumed to be imaginary. Given a finite separable extension of fields \( L/F \), we write \( \text{Nm}_{L/F} \) for the norm from \( L \) to \( F \).

If \( K \) is a non-Archimedean local field, we let \( \text{Art}_K : K^\times \xrightarrow{\sim} W_K \) be the Artin reciprocity map normalized so that uniformizers are sent to geometric Frobenius elements. For a number field \( F \), we let \( \text{Art}_F : F^\times \backslash \mathcal{A}_F^\infty \to G_F^\text{ab} \) be \( \text{Art}_F = \prod_v \text{Art}_{F_v} \). For any \( d \geq 1 \), we let \( \text{rec}_K \) be the Local Langlands reciprocity map that takes irreducible admissible representations of \( GL_d(K) \) to Frobenius semi-simple Weil–Deligne representations, normalized as in [HT01] and [Hen00]. We then let \( \text{rec}_K^T \) be given by \( \text{rec}_K^T(\pi) = \text{rec}_K(\pi \otimes |\cdot|^{1/2}) \).

If we are given an isomorphism \( \iota : K \xrightarrow{\sim} L \) of fields, and \( F \to K \) is an embedding of fields, we write \( \iota \lambda \) for \( \iota \circ \lambda \). If \( \lambda = (\lambda_i)_{i \in I} \) is a tuple of field embeddings \( F_i \to K \), then we write \( \iota \lambda \) for the tuple \((\iota \lambda_i)_{i \in I} \). If \( r : G \to \text{Aut}_K(V) \) is a representation of a group \( G \) on a \( K \)-vector space \( V \), then we will denote by \( r \lambda \) the representation of \( G \) on the \( L \)-vector space \( V \otimes_K L \).

If \( G \) is a group, \( A \) is a commutative ring, and \( \rho : G \to \text{GL}_n(A) \) is a homomorphism, then we will let \( V_\rho \) denote the representation space of \( \rho \), i.e. \( V_\rho = A^n \) with the \( A[G] \)-module structure coming from \( \rho \).
We denote by \( \ell \) the \( p \)-adic cyclotomic character. We let \( B_{ct}, B_{st}, \) and \( B_{dR} \) denote Fontaine’s rings of crystalline, semistable, and deRham periods, respectively. We will frequently use the Berger–André–Kedlaya–Mebkhout Theorem [Ber02, Théorème 0.7], that deRham representation are potentially semistable, without comment. We use covariant \( p \)-adic Hodge theory, and normalize our Hodge–Tate weights so that the Hodge–Tate weight of \( \ell \) is \(-1\). If \( K \) and \( E \) are two algebraic extensions of \( \mathbb{Q}_p \) with \( K/\mathbb{Q}_p \) finite, \( \tau : K \rightarrow E \) is a continuous embedding, and \( \rho : G_K \rightarrow \text{GL}(V) \cong \text{GL}_d(E) \) is a continuous deRham representation, we will write \( \text{HT}_\tau(\rho) \) for the multiset of \( d \)-Hodge–Tate weights with respect to \( \tau \). Specifically, an integer \( i \) appears in \( \text{HT}_\tau(\rho) \) with multiplicity equal to the \( E \)-dimension of the \( i \)th graded piece of the \( d \)-dimensional filtered -vector space \( D_{dR}(\rho) \otimes_{(K \otimes_{\mathbb{Q}_p} E)} E \), where \( D_{dR}(\rho) = (B_{dR} \otimes_{\mathbb{Q}_p} V_\rho)^{G_K} \) and we view \( E \) as a \( K \otimes_{\mathbb{Q}_p} E \)-algebra via \( \tau \otimes 1 \). We will say a continuous representation \( \rho : G_F \rightarrow \text{GL}(V) \cong \text{GL}_d(E) \) of the Galois group of a number field \( F \) is deRham, resp. semistable, resp. crystalline, if \( \rho|_{G_{F,cr}} \) is so for every \( v|p \in F \).

If \( A \) is a commutative local ring, we will denote by \( m_A \) its maximal ideal. If \( A \) is a commutative ring and \( x : A \rightarrow D \) is a homomorphism with \( D \) a domain, then we denote by \( A_x \) the localization of \( A \) at \( \ker(x) \), and \( A_x^\wedge \) the localization and completion of \( A \) at \( \ker(x) \). If \( A \) is a commutative ring and \( x \in \text{Spec} A \) has residue field \( k_x \), we again denote by \( x \) the map \( x : A \rightarrow k_x \).

Recall that if \( k \) is a field, \( R \) is a commutative \( k \)-algebra, and \( x \in \text{Spec} R \), we say \( R \) is formally smooth over \( k \) at \( x \) if there is a open subset \( U \subseteq \text{Spec} R \) such that \( k \rightarrow R_y \) is formally smooth for all \( y \in U \). If \( R \) is Noetherian and \( k \) has characteristic \( 0 \), this is equivalent to \( R_y \) being regular for all \( y \in U \). If \( R \) is further regular, the excellent locus is open, so \( R \) is formally smooth at \( x \) if and only if \( R_y \) is regular, which happens if and only if \( k \rightarrow R_y \) is formally smooth, which happens if and only if \( R_y^\wedge \) is isomorphic to a power series ring over its residue field. We will use these equivalences without further comment.

If \( B \) is a local commutative ring, we let \( \text{CNL}_B \) be the category of complete local commutative \( B \)-algebras \( A \) such that the structure map \( B \rightarrow A \) induces an isomorphism \( B/m_B \rightarrow A/m_A \), and whose morphisms are \( B \)-algebra morphisms. We will refer to an object, resp. a morphism, in \( \text{CNL}_B \) as a \( \text{CNL}_B \)-algebra, resp. a \( \text{CNL}_B \)-morphism. The full subcategory of Artinian objects is denoted \( \text{Ar}_B \).

If \( G \) is a topological group, and \( M \) is a topological \( G \)-module, the cohomology groups \( H^i(G, W) \) are always assumed to be the continuous cohomology groups, i.e. the cohomology groups computed with continuous cochains. If \( M/L \) is a Galois extension, and \( W \) is a topological Gal\( (M/L) \)-module, we write \( H^i(M/L, W) \) for \( H^i(\text{Gal}(M/L), W) \). If \( \overline{L} \) is an algebraic closure of \( L \), we write \( H^i(L, W) \) for \( H^i(\overline{L}/L, W) \). If \( K \) is a finite extension of \( \mathbb{Q}_p \), and \( W \) is a finite dimensional \( \mathbb{Q}_p \)-vector space with a continuous \( \mathbb{Q}_p \)-linear \( G_K \)-action, we set

\[
H^1_{\text{dR}}(K, W) = \ker(H^1(K, W) \rightarrow H^1(K, B_{dR} \otimes_{\mathbb{Q}_p} W)).
\]

If \( F \) is a number field, \( M \) is Galois extension of \( F \), and \( W \) is a finite dimensional \( \mathbb{Q}_p \)-vector space with a continuous \( \mathbb{Q}_p \)-linear \( \text{Gal}(M/F) \)-action, we set

\[
H^1_{\text{dR}}(M/F, W) = \ker(H^1(M/F, W) \rightarrow \prod_{v|p} H^1(F_v, B_{dR} \otimes_{\mathbb{Q}_p} W)).
\]

1. Deformation theory

Throughout this section \( E \) will denote a finite extension of \( \mathbb{Q}_p \) with ring of integers \( \mathcal{O} \) and residue field \( \mathbb{F} \).

1.1. Weil–Deligne representations. Let \( \ell \) be a rational prime, and let \( K \) be a finite extension of \( \mathbb{Q}_l \) with ring of integers \( \mathcal{O}_K \) and uniformizer \( \varpi_K \). Let \( q \) denote the cardinality of the residue field of \( K \), Let \( |\cdot| \) denote the absolute value on \( K \) normalized so that \( |\varpi_K| = q^{-1} \). For \( w \in W_K \) we will write \( |w| \) for \( |\text{Art}^{-1}(w)| \). If \( k \) is a field, and \( \rho : G_K \rightarrow \text{GL}(k) \) is a homomorphism, we let \( \text{ad}(\rho) \) denote \( \text{gl}_d(k) \) with the adjoint action \( \text{ad} \circ \rho \) of \( G_K \). Note that \( \text{ad}(\rho) \cong \text{Hom}_k(V_\rho, V_\rho) \) and \( \text{ad}(\rho)(1) \cong \text{Hom}_k(V_\rho, V_\rho(1)) \) as \( \mathbb{K}[G_K] \)-modules.

1.1.1. We recall some basics of Weil–Deligne representations (see [Tat77, §4]). Given a characteristic \( 0 \) field \( \Omega \), a Weil–Deligne representation over \( \Omega \) is a pair \((r, N)\), where \( r : W_K \rightarrow \text{GL}(V) \cong \text{GL}_d(\Omega) \) is a representation of \( W_K \) on a finite dimensional \( \Omega \)-vector space \( V \) with open kernel, and \( N \in \text{End}_{\Omega}(V) \) is nilpotent, such that \( (\lambda w) N r(w)^{-1} = |w| N \) for all \( w \in W_K \). A morphism of Weil–Deligne representations \((r_1, N_1) \rightarrow (r_2, N_2)\) is an \( \Omega \)-linear morphism that intertwines the \( r_1 \) and the \( N_1 \). A Weil–Deligne representation \((r, N)\) is called Frobenius-semisimple if \( r \) is semisimple (equivalently, if \( r(\Phi) \) is semisimple for \( \Phi \in W_K \) a lift of the Frobenius). Given a Weil–Deligne representation \((r, N)\), we will denote by \((r, N)^{F\text{-ss}}\) its Frobenius-semisimplification, i.e. \((r, N)^{F\text{-ss}} = (r^{ss}, N)\). Given a Weil–Deligne representation \((r, N)\), we
Let \((r, N)(1) = (r(1), N)\) be the Weil–Deligne representation with \(r(1)(w) = |w|r(w)\). If \(\iota \in \text{Aut}(\Omega)\), we let \(\iota(r, N) = (r, \iota N)\) denote the Weil–Deligne representation obtained by change of scalars via \(\iota: \Omega \xrightarrow{\sim} \Omega\) (this is again a Weil–Deligne representation since \(|w| \in \mathbb{Q}\) for all \(w \in W_K\).

**Definition 1.1.2.** We say a Weil–Deligne representation \((r, N)\) is **generic** if there is no nontrivial morphism \((r, N) \to (r(1), N)\).

If \(\pi\) is an irreducible admissible representation of \(G_{\mathbb{A}}(K)\) over \(\mathbb{C}\) and \(\iota \in \text{Aut}(\mathbb{C})\), then \(\text{rec}_K^T(\iota \pi) = \iota \text{rec}_K^T(\pi)\) (this is explained when \(d = 2\) in [BH06, §35] and the argument there generalizes using [BH00, Theorem 3.2] and the converse theorems of [Hen93]). If \(\Omega\) is algebraically closed with cardinality continuum, we get a bijection, again denoted by \(\text{rec}_K^T\), between isomorphism classes of irreducible admissible representations of \(G_{\mathbb{A}}(K)\) over \(\Omega\) and isomorphism classes of \(d\)-dimensional Frobenius semi-simple Weil–Deligne representations over \(\Omega\) by fixing any isomorphism \(\iota: \Omega \xrightarrow{\sim} \mathbb{C}\) and setting \(\text{rec}_K^T(\pi) = \iota^{-1}\text{rec}_K^T(\iota \pi)\), and this is independent of the choice of \(\iota\).

**Lemma 1.1.3.** Let \(\pi\) be an irreducible smooth admissible representation of \(G_{\mathbb{A}}(K)\) on an \(\Omega\)-vector space, with \(\Omega\) an algebraically closed field of characteristic 0 and continuum cardinality. Then \(\text{rec}_K^T(\pi)\) is generic if and only if \(\pi\) is generic.

**Proof.** This is essentially identical to [BLGGT14, Lemma 1.3.2(1)]. We give the details. It suffices to consider the case \(\Omega = \mathbb{C}\). Since \(\pi\) is generic if and only if \(\pi \otimes |\cdot|^{1/2}\) is generic, it is equivalent to show that \(\pi\) is generic if and only if \(\text{rec}_K^T(\pi)\) is generic. Note that if \((r, N) = \text{rec}_K^T(\pi)\), then \((r(1), N) = \text{rec}_K^T(\pi \otimes |\cdot|)\).

We will use the notation and terminology of [HT01, §1.3]. There are positive integers \(s_i, d_i\) for \(i = 1, \ldots, t\) with \(d = d_1 s_1 + \cdots + d_t s_t\) and irreducible supercuspidal representations \(\pi_i\) of \(G_{\mathbb{A}}(K)\) such that

\[
\pi \cong \text{Sp}_{s_1}(\pi_1) \oplus \cdots \oplus \text{Sp}_{s_t}(\pi_t),
\]

and the multiset \(\{(s_1, \pi_1), \ldots, (s_t, \pi_t)\}\) is uniquely determined by \(\pi\). By abuse of notation, we also denote by \(\text{Sp}_s\) the \(s\)-dimensional Weil–Deligne representation \((r, N)\) on a complex vector space with basis \(e_0, \ldots, e_{s-1}\), where \(r(w) = |w|^s e_i\) for each \(i = 0, \ldots, s-1\), and \(N e_i = e_{i+1}\) for each \(i = 0, \ldots, s-2\) and \(N e_{s-1} = 0\). Then (see [HT01, Theorem VII.2.20] and the discussion preceding it)

\[
\text{rec}_K^T(\pi) = (\text{rec}_K^T(\pi_1) \otimes \text{Sp}_{s_1}) \oplus \cdots \oplus (\text{rec}_K^T(\pi_t) \otimes \text{Sp}_{s_t}),
\]

and \(\text{rec}_K^T(\pi)\) is non-generic if and only if

\[
\text{Hom}_{\text{WD}}(\text{rec}_K^T(\pi_i) \otimes \text{Sp}_{s_i}, \text{rec}_K^T(\pi_j \otimes |\cdot|) \otimes \text{Sp}_{s_j}) \neq 0.
\]

for some \(i, j\). Since \(\text{rec}_K^T(\pi_i)\) is absolutely irreducible (as \(\pi_i\) is supercuspidal), it is easy to check that (1) holds if and only if \(s_i \geq s_j \geq a \leq s_j\). In the notation and terminology [Zel80] (see [Zel80, §3.1 and §4.1]), this happens if and only if the segments \([\pi_i, \ldots, \pi_i \otimes |\cdot|^{a-1}]\) and \([\pi_j, \ldots, \pi_j \otimes |\cdot|^{a-1}]\) are linked, which happens if and only if \(\pi\) is non-generic by [Zel80, Theorem 9.7] (note **generic** is called **non-degenerate** in [Zel80]).

1.1.4. Assume that \(\ell \neq p\), and let

\[
\rho: G_K \to \text{GL}_d(E)
\]

be a continuous representation. Following [Tat77, §4.2], we can attach a Weil–Deligne representation to our fixed \(\rho\), that we will denote \(\text{WD}(\rho)\), as follows. Fix \(\Phi \in G_K\) mapping to the geometric Frobenius in \(G_K/I_K\). Fix a surjection \(t_p: I_K \to \mathbb{Z}_p\), and let \(\tau_p \in I_K\) be such that \(t_p(\tau_p) = 1\). The homomorphism \(t_p\) necessarily factors through tame inertia. Write \(\rho(\tau_p) = \rho(\tau_p)^e\rho(\tau_p)^s\) with \(\rho(\tau_p)^e\) semisimple and \(\rho(\tau_p)^s\) unipotent. Set \(N = \log(\rho(\tau_p)^s)\). Then the map \(r: W_K \to \text{GL}_d(E)\) given by

\[
r(\Phi^n)^s = \rho(\Phi^n)^s e^{-t_p(\sigma)^N}
\]

for \(n \in \mathbb{Z}\) and \(\sigma \in I_K\), is well-defined with open kernel, and \((r, N)\) is a Weil–Deligne representation. The isomorphism class does not depend on the choices made, and we denote any element in this isomorphism class by \(\text{WD}(\rho)\). Moreover, this assignment (which depends on \(\Phi\) and \(t_p\)) gives an equivalence of categories from the category of continuous representations \(\rho: G_K \to \text{GL}_d(E)\) to the full subcategory of Weil–Deligne representations \((r, N)\) on \(E^d\) such that \(r\) has bounded image. From this we deduce the following lemma.

**Lemma 1.1.5.** Let \(\rho: G_K \to \text{GL}_d(E)\) be a continuous representation. The Weil–Deligne representation \(\text{WD}(\rho)\) is generic if and only if \(\text{Hom}_{E[G_K]}(V_{\rho}, V_{\rho}(1)) = 0\).
1.1.6. Assume $\ell = p$, and let

$$\rho : G_K \to \text{GL}_d(E)$$

be a continuous potentially semistable representation. Following Fontaine, [Fon94b, §1.3 and §2.3], we can also associate a Weil–Deligne representation to $\rho$, again denoted $\text{WD}(\rho)$, as follows.

Let $L/K$ be a finite extension, let $G_{L/K} = \text{Gal}(L/K)$, and let $L_0$ be the maximal subfield of $L$ unramified over $\mathbb{Q}_p$. We assume that $E$ contains all embeddings of $L_0$ into an algebraic closure of $E$. A $(\varphi, N, G_{L/K})$-module $D$ over $E$ is a finite free $L_0 \otimes_{\mathbb{Q}_p} E$-module together with operators $\varphi$ and $N$, and an action of $\text{Gal}(L/K)$, satisfying the following:

- $N$ is $L_0 \otimes_{\mathbb{Q}_p} E$-linear;
- $\varphi$ is $E$-linear and satisfies $\varphi(ax) = \sigma(a)\varphi(x)$ for any $x \in D$ and $a \in L_0$, where $\sigma \in \text{Gal}(L_0/\mathbb{Q}_p)$ is the absolute arithmetic Frobenius;
- $N\varphi = p\varphi N$;
- the $\text{Gal}(L/K)$-action is $E$-linear and $L_0$-semilinear, and commutes with $\varphi$ and $N$.

Extend the action of $\text{Gal}(L/K)$ to $W_K$ by letting $I_L$ act trivially. For $w \in W_K$, we let $v(w) \in \mathbb{Z}$ be such that the image of $w$ in $W_K/I_K$ is $\sigma^{-v(w)}$. We then define an $L_0 \otimes_{\mathbb{Q}_p} E$-linear action, that we denote $r_D$, of $W_K$ on $D$ by $r_D(w) = w\varphi^{v(w)}$. Writing $L_0 \otimes_{\mathbb{Q}_p} E = \prod_{\tau : L_0 \to E} E$, we get a decomposition $D = \prod_{\tau : L_0 \to E} D_\tau$ and an induced $d$-dimensional Weil–Deligne representation $(r_\tau, N_\tau)$ over $E$ on each factor $D_\tau$. The isomorphism class of $(r_\tau, N_\tau)$ is independent of $\tau : L_0 \to E$ (see [BM02, §2.2.1]), and we denote any element in its isomorphism class by $\text{WD}(D)$. Moreover, by [BS07, Proposition 4.1], this assignment induces an equivalence of categories from $(\varphi, N, G_{L/K})$-modules over $E$ to Weil–Deligne representations over $E$ on which $I_L$ acts trivially. Given a $(\varphi, N, G_{L/K})$-module $D$ over $E$, we let $D(1)$ be the $(\varphi, N, G_{L/K})$-module with the same underlying $L_0 \otimes_{\mathbb{Q}_p} E$-module, operator $N$, and $G_{L/K}$-action, but with $\varphi_{D(1)} = p^{-1}\varphi_D$. Note that $\text{WD}(D(1)) = \text{WD}(D)(1)$.

Now choose $L/K$ such that $\rho|_{G_L}$ is semistable and such that $E$ contains all embeddings of $L$ into an algebraic closure of $E$ (enlarging $E$ if necessary). We get a $(\varphi, N, G_{L/K})$-module

$$D_{st,L}(\rho) := (B_{st} \otimes_{\mathbb{Q}_p} V_\rho)^{G_L}.$$

The isomorphism class of $\text{WD}(D_{st,L}(\rho))$ does not depend on the choice of $L$ (see [BM02, §2.2.1]), and we set $\text{WD}(\rho) = \text{WD}(D_{st,L}(\rho))$.

**Lemma 1.1.7.** Let $\rho : G_K \to \text{GL}_d(E)$ be a potentially semistable representation. Let $L/K$ be a finite extension such that $\rho|_{G_L}$ is semistable.

1. The Weil–Deligne representation $\text{WD}(\rho)$ is generic if and only if the only morphism $D_{st,L}(\rho) \to D_{st,L}(\rho)(1)$ of $(\varphi, N, G_{L/K})$-modules over $E$ is the trivial one.

2. Let $D_{cr}(\text{ad}(\rho)(1)) = (B_{cr} \otimes_{\mathbb{Q}_p} \text{ad}(\rho)(1))^G_K$ with its induced crystalline Frobenius $\varphi$. Then $\text{WD}(\rho)$ is generic if and only if $D_{cr}(\text{ad}(\rho)(1))^{\varphi = 1} = 0$.

**Proof.** Part 1. follows from [BS07, Proposition 4.1]. We use this to derive part 2. By [Fon94a, §5.6], the $(\varphi, N, G_{L/K})$-module $D_{st,L}(\text{Hom}_{\mathbb{Q}_p}(V_\rho, V_\rho(1)))$ over $\mathbb{Q}_p$ is identified with the $(\varphi, N, G_{L/K})$-module over $\mathbb{Q}_p$ consisting of the $L_0$-vector space of morphisms $D_{st,L}(\rho) \to D_{st,L}(\rho(1))$ with $(\varphi, N, G_{L/K})$-module structure by

- $\varphi f = \varphi \circ f \circ \varphi^{-1}$,
- $N f = N \circ f - f \circ N$,
- $\gamma f = \gamma \circ f \circ \gamma^{-1}$, for $\gamma \in G_{L/K}$.

This identification takes the subspace of elements that commute with $E$ to the subspace of elements that commute with $E$, and we have an isomorphism $D_{st,L}(\text{ad}(\rho)(1))$ with the space of $L_0 \otimes_{\mathbb{Q}_p} E$-morphisms $D_{st,L}(\rho) \to D_{st,L}(\rho(1))$ with the $(\varphi, N, G_{L/K})$-module structure as above. This together with part 1. implies that $\text{WD}(\rho)$ is generic if and only if

$$\{ f \in D_{st,L}(\text{ad}(\rho)(1))^{G_{L/K}} \mid N f = 0 \text{ and } \varphi f = f \} = 0.$$

The left hand side of this expression is exactly the subspace of $D_{cr}(\text{ad}(\rho)(1))$ on which $\varphi = 1$. \qed

We note that one may have $\text{Hom}_{E[G_{Kc}]}(V_\rho, V_\rho(1)) = 0$, but $\text{WD}(\rho)$ nongeneric, for example if $\rho$ is a nonsplit crystalline extension of the trivial character by the cyclotomic character.
1.2. Local Galois deformation rings. We keep the notation and terminology of the previous subsection. Fix a continuous representation \( \overline{\rho} : G_K \rightarrow GL_d(\mathbb{F}) \).

A lift of \( \overline{\rho} \) to a CNL\( _d \)-algebra \( A \) is a continuous homomorphism 

\[
\rho : G_K \rightarrow GL_d(A)
\]

such that \( \rho \mod m_A = \overline{\rho} \). The set valued functor that sends a CNL\( _d \)-algebra to its set of lifts is representable (see [Böc, Proposition 1.3]). We call the representing object the universal lifting ring for \( \overline{\rho} \) and denote it by \( R^\square_{\overline{\rho}} \). We let \( \rho^\square : G_K \rightarrow GL_d(R^\square_{\overline{\rho}}) \) denote the universal lift.

In what follows, if \( R \) is a quotient of \( R^\square_{\overline{\rho}} \), and \( x \in \text{Spec} R[1/p] \) has residue field \( k \), we let \( \rho_x : G_K \rightarrow GL_d(k) \) denote the specialization of \( \rho^\square \) via \( R^\square[1/p] \rightarrow R[1/p] \xrightarrow{\bar{\rho}} k \). We then define a lift of \( \rho_x \) to a CNL\( _k \)-algebra \( A \) to be a homomorphism 

\[
\rho : G_K \rightarrow GL_d(A)
\]

such that \( \rho \mod m_A = \rho_x \) and such that the induced map \( G_K \rightarrow GL_d(A/m^n_A) \) is continuous for all \( n \geq 1 \), where we give \( A/m^n_A \) the topology as a finite dimensional \( k \)-vector space.

The proof of our main theorems will rely crucially on Kisin’s method for analyzing the generic fibre of universal deformation rings, the linchpin of which is the following result.

Theorem 1.2.1. Let \( x \) be a closed point of \( R^\square_{\overline{\rho}}[1/p] \) with residue field \( k \).

1. The set valued functor that sends a CNL\( _k \)-algebra to the set of lifts of \( \rho_x \) is represented by the the localization and completion \( (R^\square_{\overline{\rho}})^{\wedge}_x \) of \( R^\square_{\overline{\rho}} \) at \( x \).

2. The tangent space of \( \text{Spec} R^\square_{\overline{\rho}}[1/p] \) at \( x \) is canonically isomorphic to the space of 1-cocyles \( Z^1(K, ad(\rho_x)) \) of \( G_K \) with coefficients in \( ad(\rho_x) \).

Proof. Part 1. is [Kis09a, Lemma 2.3.3 and Proposition 2.3.5]. In fact, [Kis09a, Proposition 2.3.5] goes further by identifying certain groupoids, which implies what we want (see [Kis09a, §A.5]).

Using part 1., it is straightforward to check that the map \( Z^1(K, ad(\rho_x)) \rightarrow \text{Hom}_{\text{CNL}_k}((R^\square_{\overline{\rho}})^{\wedge}_x, k[\varepsilon]) \) given by \( \kappa \mapsto (1 + \varepsilon\kappa)\rho_x \) is an isomorphism of \( k \)-vector spaces. \( \square \)

Proposition 1.2.2. Assume \( \ell \neq p \).

1. \( \text{Spec} R^\square_{\overline{\rho}}[1/p] \) is equidimensional of dimension \( d^2 \).

2. A closed point \( x \) of \( \text{Spec} R^\square_{\overline{\rho}}[1/p] \) is smooth if and only if \( \text{WD}(\rho_x) \) is generic.

Proof. The fact that \( \text{Spec} R^\square_{\overline{\rho}}[1/p] \) has dimension \( d^2 \) is a result of Gee [Gee11, Theorem 2.1.6] (see also the discussion preceding Proposition 2.1.4 of [Gee11]). Let \( k \) denote the residue field of \( x \). Then 1.2.1 implies that \( x \) is a smooth point if and only if \( \dim_k Z^1(K, ad(\rho_x)) = d^2 \). By local Euler characteristic,

\[
\dim_k Z^1(K, ad(\rho_x)) = \dim_k H^1(K, ad(\rho_x)) + d^2 - \dim_k H^0(K, ad(\rho_x)) = d^2 + \dim_k H^2(K, ad(\rho_x)),
\]

so \( (R^\square_{\overline{\rho}})^{\wedge}_x \) is formally smooth over \( k \) if and only if \( H^2(K, ad(\rho_x)) = 0 \). The trace pairing on \( ad(\rho_x) \) is perfect, so Tate local duality implies that \( H^2(K, ad(\rho_x)) = 0 \) if and only if \( H^0(K, ad(\rho_x)(1)) = 0 \). This is equivalent to \( \text{Hom}_{k[G_K]}(V_{\rho_x}, V_{\rho_x}(1)) = 0 \), which is equivalent to \( \text{WD}(\rho_x) \) being generic by 1.1.5. \( \square \)

1.2.3. Assume that \( \ell = p \). An \( d \)-dimensional Galois type over \( E \) is a representation \( \tau : I_K \rightarrow GL(V) \cong GL_d(E) \) of \( I_K \) on a \( d \)-dimensional \( E \)-vector space \( V \) with open kernel that extends to a representation of \( W_K \).

An \( d \)-dimensional \( p \)-adic Hodge type over \( E \) is a pair \( \nu = (D, \{Fil_i\}_{i \in \mathbb{Z}}) \), where \( D \) is a free \( K \otimes_{\overline{Q}_p} E \)-module of rank \( d \), and \( \{Fil_i\}_{i \in \mathbb{Z}} \) is a decreasing, separated, exhaustive filtration on \( D \) by \( K \otimes_{\overline{Q}_p} E \)-submodules. We set \( \text{ad}(D) = \text{End}_{K[G_K]}(D) \) and \( \text{ad}(D)^+ = \{ f \in \text{ad}(D) \mid f(Fil_i) \subseteq Fil_{i+1} \text{ for all } i \in \mathbb{Z} \} \). We will say that a \( p \)-adic Hodge type \( \nu = (D, \{Fil_i\}_{i \in \mathbb{Z}}) \) over \( E \) is regular if for any minimal prime \( q \) of \( K \otimes_{\overline{Q}_p} E \), the \( d \)-dimensional filtered \( (K \otimes_{\overline{Q}_p} E)/q \)-vector space \( D/q \) has graded pieces of dimension at most 1. It is straightforward to check that if \( \nu \) is regular, then \( \dim E \text{ad}(D)/\text{ad}(D)^+ = \frac{d(d-1)}{2} \), and this is maximal.

Let \( \tau : I_K \rightarrow GL(V) \) and \( \nu = (D, \{Fil_i\}_{i \in \mathbb{Z}}) \) be a \( d \)-dimensional Galois type and \( p \)-adic Hodge type, respectively, over \( E \). Let \( A \) be a finite \( E \)-algebra and let \( V_A \) be a free \( A \)-module of rank \( d \) with a continuous \( A \)-linear \( G_K \)-action such that \( V_A \) is a potentially semistable representation. Let \((r_A, N_A)\) be the Weil–Deligne
representation attached to $V_A$ (viewed as a representation of $G_K$ on a $d(\dim_E A)$-dimensional $E$-vector space). We say that $V_A$ has Galois type $\tau$ if $r_A|_{K} \cong \tau \otimes_E A$. Let $D_{\text{dr}}(V_A) = (B_{\text{dr}} \otimes_{Q_p} V_A)^{G_K}$ together with its natural filtration induced from the filtration on $B_{\text{dR}}$. We say that $V_A$ has $p$-adic Hodge type $\mathbf{v}$ if for each $i \in \mathbb{Z}$, there is an isomorphism of $K \otimes_{Q_p} A$-modules

$$\text{gr}^i D_{\text{dr}}(V_A) \cong \text{gr}^i(D) \otimes_E A.$$

We can now state the following fundamental result of Kisin, [Kis08, Theorem 3.3.4].

**Theorem 1.2.4.** Fix a $d$-dimensional Galois type $\tau$, and a $d$-dimensional $p$-adic Hodge type $\mathbf{v} = (D, \{\text{Fil}^i\}_{i \in \mathbb{Z}})$ over $E$. There is an $\mathcal{O}$-flat quotient $R_{\mathcal{O}}^\square(\tau, \mathbf{v})$ of $R_{\mathcal{O}}^\square$ such that if $A$ is any finite $E$-algebra, an $E$-algebra morphism $x : R_{\mathcal{O}}^\square[1/p] \to A$ factors through $R_{\mathcal{O}}^\square(\tau, \mathbf{v})[1/p]$ if and only if $\rho_x$ is potentially semistable with Galois type $\tau$ and $p$-adic Hodge type $\mathbf{v}$.

Moreover, if nonzero, then $\text{Spec } R_{\mathcal{O}}^\square(\tau, \mathbf{v})[1/p]$ is equidimensional of dimension $d^2 + \dim_E \text{ad}(D)/\text{ad}(D)^+$, and admits an open dense formally smooth subscheme.

We now wish to show the analogue of part 2. of 1.2.2 for the rings $R_{\mathcal{O}}^\square(\tau, \mathbf{v})$. For global applications, we will actually only need the fact that $\text{WD}(\rho_x)$ generic implies $R_{\mathcal{O}}^\square(\tau, \mathbf{v})^{\square}$ is formally smooth, but for completeness we include the converse. Our proof will rely on the following standard lemma, which we will also need for other purposes later.

**Lemma 1.2.5.** Let $x$ be a closed point of $\text{Spec } R_{\mathcal{O}}^\square(\tau, \mathbf{v})[1/p]$. The tangent space of $R_{\mathcal{O}}^\square(\tau, \mathbf{v})[1/p]$ at $x$ is canonically isomorphic to

$$Z^\cdot_g(K, \text{ad}(\rho_x)) := \ker(Z^1(K, \text{ad}(\rho_x)) \to H^1(K, B_{\text{dr}} \otimes_{Q_p} \text{ad}(\rho_x))).$$

**Proof.** Let $k$ denote the residue field of $x$. Using 1.2.1, the tangent space of $(R_{\mathcal{O}}^\square)^{\square}$ is canonically isomorphic to $Z^1(K, \text{ad}(\rho_k))$. Take $\kappa \in Z^1(K, \text{ad}(\rho_k))$, and let $\rho_k = (1 + \varepsilon)\rho_k : G_K \to \text{GL}_d(k[\varepsilon])$ be the corresponding lift. The cocycle $\kappa$ dies in $H^1(K, B_{\text{dr}} \otimes_{Q_p} \text{ad}(\rho_k))$ if and only if there is a $G_K$-equivariant isomorphism

$$B_{\text{dr}} \otimes_{Q_p} V_{\rho_k} \cong B_{\text{dr}} \otimes_{Q_p} (V_{\rho_k} \otimes_k k[\varepsilon]) \cong (B_{\text{dr}} \otimes_{Q_p} V_{\rho_k}) \otimes_k k[\varepsilon],$$

and this happens if and only if $\rho_k$ is potentially semistable with $p$-adic Hodge type $\mathbf{v}$. Choosing an extension $L/K$ for which $\rho_k$ is semistable and using the exactness of $D_{st,L}$ (see [Fon94a, Théorème 5.1]), we see that the Galois type of $\rho_k$ is an extension of $\tau$ by itself. Since $\tau$ is a representation of a finite group in characteristic 0, it necessarily splits and $\rho_k$ has Galois type $\tau$. Hence, $\kappa$ lies in the kernel of $Z^1(K, \text{ad}(\rho_k)) \to H^1(K, B_{\text{dr}} \otimes_{Q_p} \text{ad}(\rho_k))$ if and only if the lift $\rho_k$ is potentially semistable of Galois type $\tau$ and $p$-adic Hodge type $\mathbf{v}$. By 1.2.4, the subspace of such elements is the tangent space of $R_{\mathcal{O}}^\square(\tau, \mathbf{v})^{\square}$.

**Lemma 1.2.6.** Let $k$ be a finite extension of $Q_p$ and let $\rho : G_K \to \text{GL}_d(k)$ be a continuous deRham representation. There is an isomorphism $D_{\text{dr}}(\rho) \cong (D_{\text{dr}}(\rho))^G_K$ of filtered $K \otimes_{Q_p} k$-modules.

**Proof.** Indeed, from $\text{ad}(\rho) \cong \text{End}_k(V_{\rho})$ and the fact that $\rho$ and $\text{ad}(\rho)$ are deRham, we have isomorphisms of filtered $K \otimes_{Q_p} k$-modules

$$D_{\text{dr}}(\rho) \cong (B_{\text{dr}} \otimes_{Q_p} \text{End}_k(V_{\rho}))^{G_K}$$

and

$$D_{\text{dr}}(\text{ad}(\rho)) \cong (B_{\text{dr}} \otimes_{Q_p} \text{End}_k(V_{\rho}))^{G_K}$$

$$\cong (B_{\text{dr}} \otimes_{Q_p} \text{End}_k(D_{\text{dr}}(\rho)))^{G_K}$$

$$\cong B_{\text{dr}}^{G_K} \otimes_k \text{ad}(D_{\text{dr}}(\rho))$$

$$= \text{ad}(D_{\text{dr}}(\rho)).$$

**Theorem 1.2.7.** A closed point $x$ of $\text{Spec } R_{\mathcal{O}}^\square(\tau, \mathbf{v})[1/p]$ is formally smooth if and only if $\text{WD}(\rho_x)$ is generic.

**Proof.** Let $k$ denote the residue field of $x$. By 1.2.4, $R_{\mathcal{O}}^\square(\tau, \mathbf{v})^{\square}$ has dimension $d^2 + \dim_E \text{ad}(D)/\text{ad}(D)^+$. Since $\rho_x$ has $p$-adic Hodge type $\mathbf{v}$, this is equal to $d^2 + \dim_k D_{\text{dr}}(\rho_x)/D_{\text{dr}}(\text{ad}(\rho_x))^+$, which equals $d^2 + \dim_k D_{\text{dr}}(\text{ad}(\rho_x))/D_{\text{dr}}(\text{ad}(\rho_x))^+$ by 1.2.6.
We now analyze the dimension of the tangent space of $R^\circ_p(\tau, \nu)^\wedge_x$. By 1.2.5, the tangent space of $R^\circ_p(\tau, \nu)^\wedge_x$ has dimension
\[ \dim_k Z^1_0(K, \text{ad}(\rho_x)) = d^2 + \dim_k H^1_0(K, \text{ad}(\rho_x)) - \dim_k H^0(K, \text{ad}(\rho_x)). \]
So, $R^\circ_p(\tau, \nu)^\wedge_x$ is smooth if and only if
\[ \dim_k H^1_0(K, \text{ad}(\rho_x)) - \dim_k H^0(K, \text{ad}(\rho_x)) = \dim_k D_{dR}(\text{ad}(\rho_x))/D_{dR}(\text{ad}(\rho_x))^+, \]
equivalently,
\[ \dim_{Q_p} H^1_0(K, \text{ad}(\rho_x)) - \dim_{Q_p} H^0(K, \text{ad}(\rho_x)) = \dim_{Q_p} D_{dR}(\text{ad}(\rho_x))/D_{dR}(\text{ad}(\rho_x))^+. \]
Before proceeding, we introduce some notation. If $W$ is a finite dimensional $Q_p$-vector space with a continuous $Q_p$-linear $G_K$-action, define the $Q_p$-vector spaces as in [BK, §3]:
\[ H^1_0(K, W) := \ker(H^1(K, W) \rightarrow H^1(K, D_{cr}^{z=1} \otimes Q_p W)), \]
\[ H^1_0(K, W) := \ker(H^1(K, W) \rightarrow H^1(K, B_{cr} \otimes Q_p W)). \]
The pairing $(X, Y) \mapsto \operatorname{tr}_{k/Q_p}(\operatorname{tr}(XY))$ is perfect on $\operatorname{ad}(\rho_x)$, so induces an isomorphism
\[ \operatorname{ad}(\rho_x)(1) \cong \operatorname{Hom}_{Q_p}(\operatorname{ad}(\rho_x), Q_p(1)). \]
Then, by [BK, Proposition 3.8],
\[ \dim_{Q_p} H^1_0(K, \text{ad}(\rho_x)) - \dim_{Q_p} H^0(K, \text{ad}(\rho_x)) \]
\[ = \dim_{Q_p} H^1(K, \text{ad}(\rho_x)) - \dim_{Q_p} H^1_0(K, \text{ad}(\rho_x)(1)) - \dim_{Q_p} H^0(K, \text{ad}(\rho_x)) \]
\[ = \dim_{Q_p} H^1(K, \text{ad}(\rho_x)) + \dim_{Q_p} H^1_0(K, \text{ad}(\rho_x)(1)) - \dim_{Q_p} H^1_0(K, \text{ad}(\rho_x)(1)) - \dim_{Q_p} H^0(K, \text{ad}(\rho_x)). \]
Using [BK, Corollary 3.8.4], this last expression equals
\[ \dim_{Q_p} D_{dR}(\text{ad}(\rho_x))/D_{dR}(\text{ad}(\rho_x))^+ + \dim_{Q_p} D_{cr}(\text{ad}(\rho_x)(1))^{e=1}. \]
Plugging this into (3), we see that $R^\circ_p(\tau, \nu)^\wedge_x$ is formally smooth if and only if $D_{cr}(\text{ad}(\rho_x)(1))^{e=1} = 0$. This happens if and only if $\operatorname{WD}(\rho_x)$ is generic by part 2. of 1.1.7. □

More thorough investigations of the smooth and singular loci in the case $d = 2$ and the case $d = 3$ and $K_0 = Q_p$ are carried out in [Kis09b, (A.1)] and [Bel14, §7]. In particular, when $d = 2$ Kisin shows in [Kis09b, (A.1)] that $\operatorname{Spec} R^\circ_p(\tau, \nu)(1/p)$ is reduced, and is either smooth or is the union of 2 smooth closed subspaces. When $d = 3$ and $K_0 = Q_p$, Bellouvin shows in [Bel14, §7.3] that $\operatorname{Spec} R^\circ_p(1, \nu)(1/p)$ is the union of 3 closed subspaces, two of which are smooth, and one of which is singular.

In practice it is often important to know that given a representation $\rho$ of $G_K$, the restriction $\rho|_{G_L}$ defines a smooth point for any finite extension $L/K$. For example, $\rho = \epsilon \otimes \chi$ with $\chi$ a finite order nontrivial character defines a smooth point on the corresponding potentially semistable deformation ring, but the restriction $\rho|_{G_L}$ to any $G_L$ that trivializes $\chi$ will not. It is not hard to see that any (mixed) pure Galois representation (i.e. one that satisfies the conclusion of the Weight–Monodromy Conjecture) will define a smooth point after any finite base change. A similar sufficient condition was noticed by Calegari [Cal12, Lemma 2.6]. However, as the following example illustrates, the condition that $\operatorname{WD}(\rho|_{G_L})$ is generic for any finite extension $L/K$ is strictly weaker than either of these.

Example 1.2.8. Choose a cocycle $\kappa$ of $G_{Q_p}$ valued in $Q_p(1)$ such that the cohomology class of $\kappa$ does not lie in $H^1_0(Q_p, Q_p(1))$. Then the representation
\[ \rho = \begin{pmatrix} 1 & \kappa \\ 1 & \epsilon^{-1} \end{pmatrix} \]
is semistable noncrystalline, and if $L/Q_p$ is any finite extension, the Weil–Deligne representation $\operatorname{WD}(\rho|_{G_L}) = (r, N)$ is given by
\[ r(\text{Frob}_L) = \begin{pmatrix} 1 & 0 \\ 0 & p^f \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 \\ \text{ } & \text{ } \end{pmatrix}, \]
where $f$ denotes the residue degree of $L$. A straightforward check shows that $\text{WD}(\rho|_{G_L})$ is generic, so letting $v$ denote the $p$-adic Hodge type of $\rho$, the restriction $\rho|_{G_L}$ defines a smooth point on $\text{Spec} \mathbb{R}^{\square}_{\mathbb{Q}_L}(1, v_L)[1/p]$ for any finite extension $L/\mathbb{Q}_p$ (where $v_L := L \otimes_K v$). Some related and more detailed computations are carried out in [Bel14, §7.3].

**Remark 1.2.9.** Let $\rho : G_K \to \text{GL}_d(E)$ a potentially semistable representation. Then part 2. of 1.1.7 shows that [BK, Proposition 3.8] and [BK, Corollary 3.8.4] imply that $G_K$ is a homomorphism, we write $\text{ad}(\rho)$ where $\rho \in \text{Hom}(G, \text{GL}_d(E))$ is a homomorphism, we write $\text{ad}(\rho)$, and $\text{ad}(\rho)(1) = 1$. So part 2. of 1.1.7 shows $\text{WD}(\rho|_{G_L})$ is generic.

We can generalize one direction of this slightly. Let $W$ be a representation of $G_K$ on a $d^2$-dimensional $E$-vector space such that it restriction to $G_L$, for some $L/K$ finite, is isomorphic to $\text{ad}(\rho)$ with $\rho : G_L \to \text{GL}_d(E)$ a potentially semistable representation. Then $\text{WD}(\rho|_{G_L})$ is generic. Indeed, the commutative diagram

$$
\begin{array}{ccc}
H^1(K, W) & \longrightarrow & H^1(K, B_{cr} \otimes_{\mathbb{Q}_p} W) \\
\downarrow & & \downarrow \\
H^1(L, W) & \longrightarrow & H^1(L, B_{cr} \otimes_{\mathbb{Q}_p} W)
\end{array}
$$

has injective vertical arrows by restriction-corestriction. It easily follows that $H^1(K, W) = H^1_L(K, W)$ if $H^1(K, W) = H^1_L(K, W)$. We mention this slight generalization because in our global applications we do not wish to restrict ourselves to CM extensions $F/F^+$ such that every $v|p$ in $F^+$ splits in $F$.

### 1.3. Global Galois deformation rings.

Throughout this subsection we assume $p > 2$.

We recall the Clozel–Harris–Taylor group scheme $\mathcal{G}_d$, which is the group scheme over $\mathbb{Z}$ defined as the semidirect product

$$(\text{GL}_d \times \text{GL}_1) \rtimes \{1, j\} = \mathcal{G}_d^0 \rtimes \{1, j\},$$

where $\mathcal{G}_d^0 = \{a \gamma^{-1}, a\}$, and the homomorphism $\nu : \mathcal{G}_d \to \text{GL}_1$ given by $\nu(g, a) = a$ and $\nu(j) = -1$. We let $\mathfrak{g}_d = \text{Lie} \mathcal{G}_d \subseteq \text{Lie} \mathcal{G}_d$, and let $\text{ad}$ denote the adjoint action of $\mathcal{G}_d$ on $\mathfrak{g}_d$, i.e

$$\text{ad}(g, a)(x) = g x g^{-1} \quad \text{and} \quad \text{ad}(j)(x) = -x.$$

If $\Gamma$ is a group, $A$ is a commutative ring and $r : \Gamma \to \mathcal{G}_d(A)$ is a homomorphism, we write $\text{ad}(r)$ for $\mathfrak{g}_d(A)$ with the adjoint action $\text{ad} \circ r$ of $\Gamma$.

The following is (part of) [CHT08, Lemma 2.1.1].

**Lemma 1.3.1.** Let $\Gamma$ be a topological group with an open subgroup $\Delta$ of index 2. Fix some $\gamma_0 \in \Gamma \setminus \Delta$. Let $A$ be a topological ring. There is a natural bijection between the following two sets.

1. Continuous homomorphisms $\omega : \Gamma \to \mathcal{G}_d(A)$ inducing an isomorphism $\Gamma/\Delta \cong \mathcal{G}_d(A)/\mathcal{G}_d^0(A)$.
2. Triples $(\rho, \mu, \langle \cdot, \cdot \rangle)$, where $\rho : \Delta \to \text{GL}_d(A)$ and $\mu : \Gamma \to A^\times$ are continuous homomorphisms and $\langle \cdot, \cdot \rangle$ is a perfect $A$-linear pairing on $A^d$ satisfying

$$\langle \rho(\delta) a, \rho(\gamma_0 \delta \gamma_0^{-1}) b \rangle = \mu(\delta) \langle a, b \rangle \quad \text{and} \quad \langle a, \rho(\gamma_0^2) b \rangle = -\mu(\gamma_0) \langle a, b \rangle$$

for all $a, b \in A^d$ and $\delta \in \Delta$.

Under this bijection, $\mu(\gamma) = (\nu \circ r)(\gamma)$ for all $\gamma \in \Gamma$, and $\langle a, b \rangle = \nu^{-1} b$ for $r(\gamma_0) = (P, \nu(\gamma_0))$. 

If $\Gamma$ is a group, $A$ is a commutative ring, $r : \Gamma \to \mathcal{G}_d(A)$ is a homomorphism, and $B$ is an $A$-algebra, we will write $r \otimes_A B$ for the composite of $r$ with the map $\mathcal{G}_d(A) \to \mathcal{G}_d(B)$. If $[r]$ is a $1 + M_d(m_A)$-conjugacy class of such homomorphisms, we will write $[r] \otimes_A B$ for the $1 + M_d(m_B)$-conjugacy class $[r \otimes_A B]$.

If $\Gamma$ is a group, $A$ is a commutative ring, $r : \Gamma \to \mathcal{G}_d(A)$ is a homomorphism, and $\Delta$ is a subgroup of $\Gamma$ such that $r(\Delta) \subseteq \mathcal{G}_d^0(A)$, we will write $r|_{\Delta}$ for the induced representation of $r$ to $\Delta$ with the projection $\mathcal{G}_d^0(A) \to \text{GL}_d(A)$. In particular, we view $A^d$ as an $A[\Delta]$-module via $r|_{\Delta}$.

We recall [CHT08, Definition 2.1.6]:

**Definition 1.3.2.** Let \( \Gamma \) be a group with index two subgroup \( \Delta \). Fix \( \gamma_0 \in \Gamma \setminus \Delta \). Let \( k \) be a field and let \( r : \Gamma \to \mathcal{G}_d(k) \) be a homomorphism with \( \Delta = r^{-1}(\mathcal{G}_d^0(k)) \). We say that \( r \) is Schur if all \( \Delta \)-irreducible subquotients of \( k^n \) are absolutely irreducible and for all \( \Delta \)-invariant subspaces \( k^n \supset W_1 \supset W_2 \) such that \( k^n/W_1 \) and \( W_2 \) are irreducible, we have \( (k^n/W_1)^{\gamma_0} \not\cong W_2^{\nu} \otimes (\nu \circ r) \).

Note that if \( r|_\Delta \) is absolutely irreducible, then \( r \) is Schur.

1.3.3. Before proceeding with deformation theory, we prove some results on the cohomology of the adjoint representation valued in a finite extension of \( \mathbb{Q}_p \). Let \( F \) be a CM field with maximal totally real subfield \( F^+ \). Let \( S \) be a finite set of finite places of \( F^+ \) containing all those above \( p \). Let \( k \) be a finite extension of \( \mathbb{Q}_p \), let

\[
\begin{align*}
\gamma & : \text{Gal}(F(S)/F^+) \to \mathcal{G}_d(k)
\end{align*}
\]

be a continuous homomorphism inducing an isomorphism \( \text{Gal}(F/F^+) \cong \mathcal{G}_d(k)/\mathcal{G}_d^0(k) \), and let \( \mu = \nu \circ r \).

For each \( v \infty \) in \( F \), let \( c_v \in G_{F^+} \) be a choice of complex conjugation. Recall we have assumed \( p > 2 \).

**Lemma 1.3.4.** Let the notation and assumptions be as in 1.3.3 above. Then

\[
\sum_{i=0}^{2} \dim_k H^i(F(S)/F^+, \text{ad}(r)) = -d^2[F^+: \mathbb{Q}] + \sum_{v|\infty} \frac{d(d+\mu(c_v))}{2}.
\]

**Proof.** An easy computation (see [CHT08, Lemma 2.1.3]) shows \( \dim_k H^0(F_v^+, \text{ad}(r)) = \frac{d(d+\mu(c_v))}{2} \) for each \( v \infty \). Using [CHT08, Lemma 2.1.5], we may assume \( r \) takes values in \( \mathcal{G}_d(O_k) \), where \( O_k \) is the ring of integers of \( k \). The lemma now follows from [CHT08, Lemma 2.3.3] by an argument as in [Kis03, Lemma 9.7].

\( \square \)

**Lemma 1.3.5.** Let the assumptions and notation be as in 1.3.3 above. Assume further that \( r|_{G_\infty} \) is deRham with regular \( p \)-adic Hodge type for every \( v \infty \) in \( F \), that \( \mu(c_v) = -1 \) for every \( v \infty \), and that \( \text{ad}(r)|_{G_{F^+}} = 0 \). If \( H^1_{\text{dR}}(F(S)/F^+, \text{ad}(r)) = 0 \), then the following hold.

1. \( \dim_k H^1(F(S)/F^+, \text{ad}(r)) = \frac{d(d+1)}{2}[F^+: \mathbb{Q}] \).

2. \( H^2(F(S)/F^+, \text{ad}(r)) = 0 \).

3. The natural map

\[
H^1(F(S)/F^+, \text{ad}(r)) \to \prod_{v|p} H^1(F_v^+, \text{ad}(r))/H^1_{\text{dR}}(F_v^+, \text{ad}(r))
\]

is an isomorphism.

**Proof.** The argument is exactly as in the proof of [Kis04, Theorem 8.2]. We give the details.

Using our assumption that \( \mu(c_v) = -1 \) for every \( v \infty \), and that \( \text{ad}(r)|_{G_{F^+}} = 0 \), 1.3.4 implies

\[
\dim_k H^1(F(S)/F^+, \text{ad}(r)) - \dim_k H^2(F(S)/F^+, \text{ad}(r)) = \frac{d(d+1)}{2}[F^+: \mathbb{Q}],
\]

so parts 1. and 2. are equivalent. Using the assumption \( H^1_{\text{dR}}(F(S)/F^+, \text{ad}(r)) = 0 \), there is an injection

\[
H^1(F(S)/F^+, \text{ad}(r)) \to \prod_{v|p} H^1(F_v^+, \text{ad}(r))/H^1_{\text{dR}}(F_v^+, \text{ad}(r)),
\]

and all three parts of the lemma will follow from showing

\[
\dim_k H^1(F_v^+, \text{ad}(r))/H^1_{\text{dR}}(F_v^+, \text{ad}(r)) \leq \frac{d(d+1)}{2}[F_v^+: \mathbb{Q}_p]
\]

for each \( v|p \) in \( F^+ \). Equivalently, it suffices to show that for each \( v|p \) in \( F^+ \), the image of

\[
H^1(F_v^+, \text{ad}(r)) \to H^1(F_v^+, B_{\text{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(r))
\]

has \( k \)-dimension \( \leq \frac{d(d+1)}{2}[F_v^+: \mathbb{Q}_p] \). The \( k \)-vector space \( H^1(F_v^+, B_{\text{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(r)) \) has filtration induced by the filtration on \( B_{\text{dR}} \), and the image of \( H^1(F_v^+, \text{ad}(r)) \) is contained in the \( \text{Fil}^0 \). It thus suffices to show that for each \( v|p \) in \( F^+ \),

\[
\dim_k \text{Fil}^0 H^1(F_v^+, B_{\text{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(r)) \leq \frac{d(d+1)}{2}[F_v^+: \mathbb{Q}_p].
\]
From the filtered $G_v$-equivariant isomorphism $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(r) \cong B_{\mathrm{dR}} \otimes_{F_v^+} D_{\mathrm{dR}}(\text{ad}(r))$, and the fact that $H^1(F_v^+, B_{\mathrm{dR}}) \cong F_v^+$ (see [Tat, §3]), we have filtered isomorphisms

$$H^1(F_v^+, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(r)) \cong H^1(F_v^+, B_{\mathrm{dR}} \otimes_{F_v^+} D_{\mathrm{dR}}(\text{ad}(r)))$$

$$\cong H^1(F_v^+, B_{\mathrm{dR}}) \otimes_{F_v^+} D_{\mathrm{dR}}(\text{ad}(r))$$

$$\cong D_{\mathrm{dR}}(\text{ad}(r)).$$

So, we are reduced to showing that $\dim_k \text{Fil}^0 D_{\mathrm{dR}}(\text{ad}(r)) = \frac{d(d+1)}{2} |F_v^+ : \mathbb{Q}_p|$ for each $v | p$ in $F$. For any $w | p$ in $F$, since $r|_{G_w}$ is deRham, there is an isomorphism of filtered $F_w \otimes_{\mathbb{Q}_p} k$-modules (see 1.2.6)

$$(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(r))^{G_w} \cong \text{ad}((B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_{r|_{G_w}})^{G_w}),$$

and since $r|_{G_w}$ has regular $p$-adic Hodge type,

$$\dim_k \text{Fil}^0 \text{ad}((B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_{r|_{G_w}})^{G_w}) = \frac{d(d+1)}{2} |F_w : \mathbb{Q}_p|.$$

If $v$ splits in $F$ as $ww'$, then the choice of $w$ induces an isomorphism $F_v^+ \cong F_w$, and

$$\dim_k \text{Fil}^0 D_{\mathrm{dR}}(\text{ad}(r)) = \frac{d(d+1)}{2} |F_v^+ : \mathbb{Q}_p|.$$ 

If $v$ does not split in $F$, then letting $w$ denote the unique place dividing $v$ in $F$, we have $|F_w : F_v^+| = 2$, and there is a filtered isomorphism

$$(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(r))^{G_w} \cong F_w \otimes_{F_v^+} D_{\mathrm{dR}}(\text{ad}(r)),$$

and we have

$$\dim_k \text{Fil}^0 D_{\mathrm{dR}}(\text{ad}(r)) = \frac{1}{2} \dim_k \text{Fil}^0 (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(r))^{G_w} = \frac{d(d+1)}{2} |F_v^+ : \mathbb{Q}_p|.$$

We now recall the $G_d$-valued deformation theory of [CHT08].

**Definition 1.3.6.** Let $k$ be either a finite extension of $F_p$ or of $\mathbb{Q}_p$. Let $\Gamma$ be a topological group and let $\mathfrak{r} : \Gamma \to G_d(k)$ be a continuous homomorphism. Let $A$ be a pro-Artinian local ring with a fixed isomorphism $A/m_A \cong k$.

A lift of $\mathfrak{r}$ to $A$ is a homomorphism $r : \Gamma \to G_d(A)$ such that $r \otimes_A k = \mathfrak{r}$ and that for any Artinian quotient $A \to A'$, the homomorphism $r \otimes_A A'$ is continuous, where we give $A'$ the discrete topology if $k$ is a finite extension of $\mathbb{F}_p$, and the topology as a finite dimensional $k$-vector space if $k$ is a finite extension of $\mathbb{Q}_p$. A deformation of $\mathfrak{r}$ to $A$ is a $1 + m_d(m_A)$-conjugacy class of lifts.

For a finite set $T$, a $T$-framed lift of $\mathfrak{r}$ to $A$ is a tuple $(r, \{\alpha_v\}_{v \in T})$ where $r$ is a lift of $\mathfrak{r}$ to $A$ and $\alpha_v \in \ker(GL_d(A) \to GL_d(k))$. We say two $T$-framed lifts $(r, \{\alpha_v\}_{v \in T})$ and $(r', \{\alpha'_v\}_{v \in T})$ to $A$ are equivalent if there is $g \in \ker(GL_d(A) \to GL_d(k))$ such that $g r g^{-1} = r'$ and $g \alpha_v = \alpha'_v$ for each $v \in T$. A $T$-framed deformation of $\mathfrak{r}$ to $A$ is an equivalence class of $T$-framed lifts.

If $r$ is a lift, we will write $[r]$ for the corresponding deformation. If $(r, \{\alpha_v\})$ is a $T$-framed lift, we will write $[r, \{\alpha_v\}]$ for the corresponding $T$-framed deformation.

We will introduce a slight variation of the global deformation problem of [CHT08, §2.3].

**Definition 1.3.7.** A global deformation datum is a tuple

$$\mathcal{S} = (F/F^+, S, \mathcal{S}, \mathcal{O}, \mathfrak{r}, \mu, \{R_w\}_{w \in \mathcal{S}})$$

where

- $F$ is a CM field with maximal totally real subfield $F^+$;
- $S$ is a finite set of finite places of $F^+$;
- $\mathcal{S}$ is a finite set of finite places of $F$ such that every $w \in \mathcal{S}$ is split over some $v \in S$, and $\mathcal{S}$ contains at most one place above any $v \in S$;
- $\mathcal{O}$ is the ring of integers of some finite extension of $\mathbb{Q}_p$ with residue field $\mathcal{F}$;
- $\mathfrak{r} : \text{Gal}(F(S)/F^+) \to G_d(\mathcal{F})$ is a continuous homomorphism;
- $\mu : \text{Gal}(F(S)/F^+) \to \mathcal{O}^\times$ is a continuous character with $\mu \text{ mod } m_\mathcal{O} = \nu \circ \mathfrak{r}$;
for each \( w \in \tilde{S} \), \( R_w \) is a quotient of \( R_w^\square := R_{w|G_w}^\square \) satisfying the following property: if \( \rho : G_w \to \text{GL}_d(A) \) is a lift of \( \tau|_{G_w} \) to \( A \) and \( g \in 1 + \text{M}_d(\mathfrak{m}_A) \), then the map \( R_w^\square \to A \) induced by \( \rho \) factors through \( R_w \) if and only if the map \( R_w^\square \to A \) induced by \( g\rho^{-1} \) factors through \( R_w \).

This differs from the definition in [CHT08, §2.3] in that our ramification set \( S \) may contain places that do not split in \( F/F^+ \), and \( \tilde{S} \) is not required to contain a place above every \( v \in S \). When proving modularity of Galois representations, one can use base change and descent to reduce to the case that the ramification set splits in \( F/F^+ \), and for the proof of 3.1.5, which implies Theorems A, B, and C from the introduction, it would also suffice to consider this situation because we may also use base change in its proof. But we wish to have the statement of Theorem C in the above level of generality for applications to the problem of density of automorphic points in deformation rings, where it is not obvious (at least not to the author) how to apply base change and descent.

**Definition 1.3.8.** Let \( \mathcal{S} = (F/F^+, S, \tilde{S}, \bar{O}, \tau, \mu, \{ R_w \}_{w \in \tilde{S}}) \) be a global deformation datum, and let \( A \) be a \( \text{CNL}_{\text{O}} \)-algebra. We say a lift \( r: G_{F^+} \to G_d(A) \) of \( \tau \) to \( A \) is type \( \mathcal{S} \) if

- \( r \) factors through \( \text{Gal}(F(S)/F^+) \);
- \( \nu \circ r = \mu \);
- for each \( w \in \tilde{S} \), the \( \text{CNL}_{\text{O}} \)-morphism \( R_w^\square \to A \) induced by the lift \( r|_{G_w} \) of \( \tau|_{G_w} \), factors through \( R_w \).

We say a deformation of \( \tau \) to \( A \) is type \( \mathcal{S} \) if (one equivalently any) lift in its deformation class is type \( \mathcal{S} \). We let \( D_{\emptyset} \) be the set valued functor on \( \text{CNL}_{\text{O}} \) that takes a \( \text{CNL}_{\text{O}} \)-algebra \( A \) to the set of deformations of type \( \mathcal{S} \). If \( D_{\emptyset} \) is representable, we call the representing object the *universal type \( \mathcal{S} \) deformation ring* and denote it by \( R_{\mathcal{S}} \).

For any \( T \subseteq \tilde{S} \), we say a \( T \)-framed deformation \( [r, \{ \alpha_w \}] \) of \( \tau \) to \( A \) is type \( \mathcal{S} \) if \( [r] \) is a type \( \mathcal{S} \) deformation of \( \tau \). We let \( D_{\emptyset,T} \) be the set valued functor on \( \text{CNL}_{\text{O}} \) that takes a \( \text{CNL}_{\text{O}} \)-algebra \( A \) to the set of \( T \)-framed deformations of type \( \mathcal{S} \). If \( D_{\emptyset,T} \) is representable, we call the representing object the *universal type \( \mathcal{S} \) \( T \)-framed deformation ring* and denote it by \( R_{\mathcal{S},T}^\square \). If \( T = \tilde{S} \), then we will write \( D_{\emptyset}^\square \) and \( R_{\mathcal{S}}^\square \) for \( D_{\emptyset,T}^\square \) and \( R_{\mathcal{S},T}^\square \), respectively, and call \( R_{\mathcal{S}}^\square \) the *universal type \( \mathcal{S} \) framed deformation ring*.

If \( \mathcal{S} = (F/F^+, S, \tilde{S}, \bar{O}, \tau, \mu, \{ R_w \}_{w \in \tilde{S}}) \) is a global deformation datum and \( T \subseteq \tilde{S} \), we set

\[
R_{T}^\square = \bigotimes_{w \in T} R_{w}^\square \quad \text{and} \quad R_{S,T}^\square = \bigotimes_{w \in T} R_{w}
\]

Note that \( R_{S,T}^\square \) is naturally a quotient of \( R_{T}^\square \). If \( T = \tilde{S} \), then we will write \( R_{\mathcal{S}}^\square \) for \( R_{S,T}^\square \).

The following proposition follows from [CHT08, Proposition 2.2.9].

**Proposition 1.3.9.** Let \( \mathcal{S} = (F/F^+, S, \tilde{S}, \bar{O}, \tau, \mu, \{ R_w \}_{w \in \tilde{S}}) \) be a global deformation datum, and let \( T \subseteq \tilde{S} \). Assume \( \tau \) is Schur.

The functors \( D_{\emptyset,T}^\square \) and \( D_{\emptyset} \) are representable. There is a canonical \( \text{CNL}_{\text{O}} \)-morphism \( R_{S,T}^\square \to R_{\mathcal{S}}^\square \). There is a canonical \( \text{CNL}_{\text{O}} \)-morphism \( R_{\mathcal{S}} \to R_{\mathcal{S}}^\square \), and a choice of lift

\[
r_{\mathcal{S}}^{\text{univ}} : \text{Gal}(F(S)/F^+) \to \mathcal{G}_d(R_{\mathcal{S}})
\]

for the universal type \( \mathcal{S} \) deformation \( [r_{\mathcal{S}}^{\text{univ}}] \) determines an extension of this \( \text{CNL}_{\text{O}} \)-morphism to an isomorphism \( R_{\mathcal{S}}[[X_1, \ldots, X_d]] \xrightarrow{\sim} R_{\mathcal{S}}^\square \).

1.3.10. For the remainder of this section, we fix a global deformation datum

\[
\mathcal{S} = (F/F^+, S, \tilde{S}, \bar{O}, \tau, \mu, \{ R_w \}_{w \in \tilde{S}}),
\]

with \( \tau \) Schur, where we will specify the rings \( R_w \) in the statements of the following propositions. Let \( R_{\mathcal{S}} \) be the universal type \( \mathcal{S} \) deformation ring and let \( x \) be a closed point of \( \text{Spec} R_{\mathcal{S}}[1/p] \) with residue field \( k \). Let \( O_k \) be the ring of integers of \( k \), and let \( r \) be a type \( \mathcal{S} \) lift of \( \tau \) to \( O_k \) such that the map \( R_{\mathcal{S}} \to O_k \) induced by \( [r] \) induces \( x : R_{\mathcal{S}}[1/p] \to k \). Set \( r_x = r \otimes O_k \).

**Proposition 1.3.11.** Let the notation and assumptions be as in 1.3.10 above, with \( R_w = R_w^\square \) for every \( w \in \tilde{S} \).

1. Let \( D_{\mathcal{S},r_x} \) be the functor on \( \text{Ar}_k \) that sends an \( \text{Ar}_k \)-algebra \( B \) to the set of deformations \( [r_B] \) of \( r_x \) to \( B \) that factor through \( \text{Gal}(F(S)/F^+) \) and satisfy \( \nu \circ r_B = \mu \). The functor \( D_{\mathcal{S},r_x} \) is prerepresented by \( (R_{\mathcal{S}})^{\square}_{r_x} \).
2. The tangent space of $R_S[1/p]$ at $x$ is canonically isomorphic to $H^1(F(S)/F^+, \text{ad}(r_x))$.

3. Let $S_\infty$ be the set of infinite places in $F^+$, and for very $v \in S_\infty$, let $c_v$ be a choice of complex conjugation at $v$. Then $(R_S)_x^\circ$ is isomorphic to a power series over $k$ in $g = \dim_k H^1(F(S)/F^+, \text{ad}(r_x))$ variables modulo $r$ relations with $r \leq \dim_k H^2(F(S)/F^+, \text{ad}(r_x))$, and

$$g - r \geq d^2[F^+ : \mathbb{Q}] - \sum_{v \in S_\infty} d(d + \mu(c_v)).$$

**Proof.** The proof of 1. is almost identical to that of [Kis09a, Proposition 2.3.5]. We sketch the details. We will not use the language of groupoids here, but the results we will reference from [Kis09a] stated in terms of groupoids imply our results stated in terms of functors by [Kis09a, §A.5].

Since $R_S^\circ$, for $w \in \tilde{S}$, imposes no condition on our lifts, it is easy to see that if $S'$ is the deformation datum

$$S' = (F/F^+, \mathcal{O}, \mathcal{O}, \mu, \emptyset),$$

then there is a canonical isomorphism $R_S \cong R_{S'}$. For the remainder of the proof, we assume $\tilde{S} = \emptyset$.

For any $A_k$-algebra $B$, we let $\text{Int}(B)$ be the set of all finite $\mathcal{O}$-subalgebras $A \subset B$ such that $A[1/p] = B$. Note that any $A \in \text{Int}(B)$ comes equipped with a canonical $\mathcal{O}$-algebra map $A \to \mathcal{O}_A$ via $A \subset B \to B/\mathfrak{m}_B = k$. Also note that $\text{Int}(B)$ is naturally filtered. We let $D_{S,(r)}$ be the set valued functor on $A_k$ defined by

$$D_{S,(r)}(B) = \lim_{\mathcal{C} \in \text{Int}(B)} \{[r_A] \text{ is a type } \mathcal{S} \text{ deformation of } \tau \text{ such that } [r_A] \otimes_A \mathcal{O}_k = [r]\}.$$  

By [Kis09a, Lemma 2.3.3], the functor $D_{S,(r)}$ is pro-represented by $(R_S)_x^\circ$. There is a natural morphism of functors $D_{S,(r)} \to D_{S,r_x}$ that we wish to show is an isomorphism. For any $A_k$-algebra $B$, and continuous homomorphism $r_B : \text{Gal}(F(S)/F^+) \to \text{Gal}_d(B)$, an argument as in [Kis03, Proposition 9.5] shows that there is some $A \in \text{Int}(B)$ and a continuous homomorphism $r_A : \text{Gal}(F(S)/F^+) \to \text{Gal}_d(A)$ such that $r_B = r_A \otimes_A B$. If $r_B \in D_{S,r_x}(A)$, then such an $r_A$ must by type $\mathcal{S}$, so $D_{S,(r)} \to D_{S,r_x}$ is surjective. To see that it is injective, note that for any $A_k$-algebra $B$ and any $g \in 1 + \mathfrak{m}_B$, the $\mathcal{O}$-algebra generated by the entries of $g$ is finite over $\mathcal{O}$. Thus, any two lifts of $r_B$ to an $A_k$-algebra $B$ defining the same deformation arise from lifts to some $A \in \text{Int}(B)$ that define the same deformation to $A$. This finishes the proof of part 1.

For part 2., letting $Z^1(F(S)/F^+, \text{ad}(r_x))$ be the space of continuous 1-cocycles of $\text{Gal}(F(S)/F^+)$ with values in $\text{ad}(r_x)$, the map $\kappa \mapsto r_\kappa := (1 + \kappa r_x)$ defines an isomorphism from $Z^1(F(S)/F^+, \text{ad}(r_x))$ to the $k$-vector space of lifts $r_x$ of $r_x$ to the dual numbers $k[\varepsilon]$ that factor through $\text{Gal}(F(S)/F^+)$ and satisfy $\nu \circ r_x = \mu$. Two such cocycles $\kappa$ and $\kappa'$ are cohomologous if and only if $r_\kappa$ and $r_{\kappa'}$ are conjugate by an element of $1 + \varepsilon \mathfrak{m}_d(k)$, which maps induces a canonical isomorphism from $H^1(F(S)/F^+, \text{ad}(r_x))$ to the $k$-vector space of deformations $[r_x]$ of $r_x$ to $k[\varepsilon]$ with $\nu \circ r_x = \mu$, which is isomorphic to the tangent space of $R_S[1/p]$ at $x$ by part 1.

We now show part 3. By part 2., we can fix a surjection $A := k[[X_1, \ldots, X_n]] \to (R_S)_x^\circ$ with $g = \dim_k H^1(F(S)/F^+, \text{ad}(r_x))$ that induces an isomorphism on tangent spaces. Let $J$ denote its kernel. Choose a lift $r_x^{\text{univ}} : \text{Gal}(F(S)/F^+) \to \text{Gal}_d((R_S)_x^\circ)$ of $r_x$ in the universal $(R_S)_x^\circ$-valued deformation. We view $A/m_A J$ and $(R_S)_x^\circ$ as topological rings with the topology as inverse limits of finite dimensional $k$-vector spaces. We can choose a continuous set-theoretic section $s_0 : (R_S)_x^\circ \to A/m_A J$ of the surjection $A/m_A J \to (R_S)_x^\circ$ with the property that $s_0(a) = a$ for any $a \in k$. Then $s = s_0 \circ r_x^{\text{univ}}$ is a continuous set-theoretic lift $s : \text{Gal}(F(S)/F^+) \to \text{Gal}(A/m_A J)$ of $r_x^{\text{univ}}$ with the property that $\nu \circ s(\sigma) = \mu(\sigma)$ for all $\sigma \in \text{Gal}(F(S)/F^+)$. We can then define a 2-cocycle $\kappa$ of $\text{Gal}(F(S)/F^+)$ valued in $\text{ad}(r_x) \otimes_k J/m_A J$ by

$$(\kappa(\sigma, \tau), 1) = s(\sigma) s(\tau)^{-1} s(\tau)^{-1} \in \ker(G_2^0(A/m_A J) \to G_2^0((R_S)_x^\circ)) \cong (\text{ad}(r_x) \otimes_k J/m_A J) \times (1 + J)/(1 + m_A J).$$

The cohomology class

$$[\kappa] \in H^2(F(S)/F^+, \text{ad}(r_x) \otimes_k J/m_A J) \cong H^2(F(S)/F^+, \text{ad}(r_x)) \otimes_k J/m_A J$$

does not depend on our choices. The argument of [Maz, §1.6] shows that the map $\text{Hom}_k(J/m_A J, k) \to H^2(F(S)/F^+, \text{ad}(r_x))$ given by $f \mapsto (1 \otimes f)([\kappa])$ is injective. Hence, $(R_S)_x^\circ$ is isomorphic to a power series over $k$ in $g$ variables modulo $r$ relations with $r \leq \dim_k H^2(F(S)/F^+, \text{ad}(r_x))$. Since $\tau$ is Schur, $\text{ad}(\tau)^{G_{F^+}} = 0$ (see [CHT08, Lemma 2.1.7]), which implies $\text{ad}(r_x)^{G_{F^+}} = 0$. The final claim now follows from 1.3.4. □
Proposition 1.3.12. Let the assumptions and notation be as in 1.3.10. Assume that \( \mu \) is deRham. Assume that for every \( v \mid p \) in \( F^+ \), \( v \) splits in \( F \) and \( S_p \) contains a place above \( v \), which we will denote by \( \tilde{v} \). For each \( \tilde{v} \in \tilde{S}_p \), we fix a Galois type \( \tau_0 \) and a \( p \)-adic Hodge type \( \nu_0 \) over \( E \) (see 1.2.3). Take \( R_w = R^\square_w(\tau_0, \nu_0) \) (see 1.2.4) for \( w = \tilde{v} \in \tilde{S}_p \), and \( R_w = R^\square_w \) for each \( w \in \tilde{S} \setminus \tilde{S}_p \).

1. Let \( D_{S,x} \) be the functor on \( A_k \) that sends an \( A_k \)-algebra \( B \) to the set of deformations \( \{ r_B \} \) of \( r_x \) to \( B \) that factor through \( \text{Gal}(F(S)/F^+) \), satisfy \( \nu \circ r_B = \mu \), and are such that \( r_B |_{G_S} \) is potentially semistable with Galois type \( \tau_0 \) and \( p \)-adic Hodge type \( \nu_0 \) for each \( \tilde{v} \in \tilde{S}_p \). The functor \( D_{S,x} \) is represented by \( (R_S)_{\square}^\square \).

2. The tangent space of \( R_S[1/p] \) at \( x \) is canonically isomorphic to \( H^1_S(F(S)/F^+, \text{ad}(r_x)) \).

Proof. Since \( R^\square_w \), for \( w \in \tilde{S} \setminus \tilde{S}_p \), imposes no condition on our lifts, it is easy to see that if \( S' \) is the deformation datum

\[
S' = (F/F^+, S, \tilde{S}_p, O, \tau, \mu, \{ R^\square_w(\tau_0, \nu_0) \}_{\tilde{v} \in \tilde{S}_p}),
\]

there is a canonical isomorphism \( R_S \cong R_{S'} \), and we may assume \( S = \tilde{S}_p \).

Let \( S_{\text{big}} = (F/F^+, S, \tilde{S}_p, O, \tau, \mu, \{ R^\square_w \}_{\tilde{v} \in \tilde{S}_p}) \), and let \( R_{S_{\text{big}}} \) be the universal type \( S_{\text{big}} \) deformation ring. Note \( R^\square_w \rightarrow R_{S_{\text{big}}} \). By 1.3.9, choosing a lift \( r^\square_w \) in the universal type \( S_{\text{big}} \) deformation yields a morphism \( R^\square_w \rightarrow R_{S_{\text{big}}} \). It is easy to see that \( R_S \cong R_{S_{\text{big}}} \otimes_{R^\square_w} R^\square_S \), with \( R^\square_S \rightarrow R_{S_{\text{big}}} \) the natural surjection. Using this and part 1. of 1.3.11, we see that \( (R_S)^\square \) represents the functor on \( A_k \) that sends an \( A_k \)-algebra \( B \) to the set of deformations \( \{ r_B \} \) of \( r_x \) to \( B \) that factor through \( \text{Gal}(F(S)/F^+) \), satisfy \( \nu \circ r_B = \mu \), and are such that the induced map

\[
R^\square_w \rightarrow R_{S_{\text{big}}} \rightarrow (R_{S_{\text{big}}})^\square \rightarrow B
\]

factors through \( R^\square_w(\tau_0, \nu_0) \) for each \( \tilde{v} \in \tilde{S}_p \), which happens if and only if \( r_B |_{G_S} \) is potentially semistable with Galois type \( \tau_0 \) and \( p \)-adic Hodge type \( \nu_0 \) by 1.2.4. This shows part 1.

We now show part 2. Part 1. together with part 2. of 1.3.11, implies that the tangent space of \( R_S[1/p] \) at \( x \) is canonically isomorphic to the subspace of \( H^1(F(S)/F^+, \text{ad}(r_x)) \) consisting of cohomology classes \( [\kappa] \) such that for any cocycle \( \kappa \) in the class \( [\kappa] \), the local lift

\[
r_{\kappa}|_{G_S} = ((1 + \varepsilon \kappa)r_x)|_{G_S} = (1 + \varepsilon \kappa)|_{G_S}(r_x|_{G_S})
\]

lies in the tangent space of \( R^\square_w(\tau_0, \nu_0)[1/p] \) at \( x \), for each \( \tilde{v} \in \tilde{S}_p \). By 1.2.5, this happens if and only if \( [\kappa] \) lies in the kernel of

\[
H^1(F(S)/F^+, \text{ad}(r_x)) \rightarrow \prod_{\tilde{v} \in \tilde{S}_p} H^1(F_{\tilde{v}}, B_{dR} \otimes_{Q_p} \text{ad}(r_x)).
\]

Since \( \mu \) is deRham, \( r_{\kappa}|_{G_S} \cong (r_{\kappa}|_{G_S})^c \cong (r_{\kappa}|_{G_S})^c \otimes \mu \) is deRham for each \( \tilde{v} \in \tilde{S}_p \). Thus, the tangent space of \( R_S[1/p] \) at \( x \) is canonically isomorphic to

\[
\ker \left( H^1(F(S)/F^+, \text{ad}(r_x)) \right) \rightarrow \prod_{\tilde{v} \in \tilde{S}_p} H^1(F_{\tilde{v}}, B_{dR} \otimes_{Q_p} \text{ad}(r_x))
\]

\[
= \ker \left( H^1(F(S)/F^+, \text{ad}(r_x)) \right) \rightarrow \prod_{w \mid p \text{ in } F} H^1(F_w, B_{dR} \otimes_{Q_p} \text{ad}(r_x))
\]

\[
= \ker \left( H^1(F(S)/F^+, \text{ad}(r_x)) \right) \rightarrow \prod_{w \mid p \text{ in } F^+} H^1(F_w, B_{dR} \otimes_{Q_p} \text{ad}(r_x)).
\]

We note that in 1.3.12 above the existence of \( x \in \text{Spec } R_S[1/p] \) implicitly assumes that \( R_S[1/p] \neq 0 \), which (at the very least) implies a certain compatibility between \( \mu \) and the local \( p \)-adic Hodge theory data \( \tau_0 \) and \( \nu_0 \). In our applications, we will be given a potentially semistable \( p \)-adic representation \( \rho \) and we will define \( S \) using \( \rho \). The existence of this \( \rho \) will then imply \( R_S[1/p] \neq 0 \).

Proposition 1.3.13. Let the notation and assumptions be as in 1.3.10, with \( R_w = R^\square_w \) for every \( w \in \tilde{S} \). Assume also that \( \mu(c) = -1 \) for every choice of complex conjugation \( c \in G_{F^+} \), and that \( r_x|_{G_w} : G_w \rightarrow \text{GL}_d(k) \) is deRham with regular \( p \)-adic Hodge type (see 1.2.3) for every \( w \mid p \) in \( F \).
If $H^1_q(F(S)/F^+, \text{ad}(r_\omega)) = 0$, then $(R_S)^{\gamma}_{\omega}$ is formally smooth of dimension $\frac{d(d+1)}{2}[F^+ : \mathbb{Q}]$.

Proof. Using our assumption that $\mu(c) = -1$ for every complex conjugation $c \in G_{F^+}$, part 3. of 1.3.11 implies

$$\dim(R_S)^{\gamma}_{\omega} \geq \frac{d(d+1)}{2}[F^+ : \mathbb{Q}].$$

Thus, it suffices to show the tangent space of $(R_S)^{\gamma}_{\omega}$, which is isomorphic to $H^1(F(S)/F^+, \text{ad}(r_\omega))$ by part 2. of 1.3.11, has $k$-dimension $\frac{d(d+1)}{2}[F^+ : \mathbb{Q}]$. This follows from part 1. of 1.3.5. \qed

2. Automorphic theory

This section reviews the automorphic theory that we will use to prove our main theorems. We first introduce some notation and assumptions that will be used throughout this section.

Let $\mathbb{Z}_+^d$ be the set of tuples of integers $(\lambda_1, \ldots, \lambda_d)$ such that $\lambda_1 \geq \cdots \geq \lambda_d$. If $F$ is a CM field with maximal totally real subfield $F^+$, and $\Omega$ is any characteristic 0 field containing all embeddings of $F$ into an algebraic closure of $\Omega$, then we let $(\mathbb{Z}_+^d)^{\text{Hom}(F, \Omega)}$ denote the subset of $(\mathbb{Z}_+^d)^{\text{Hom}(F, \Omega)}$ of tuples $(\lambda_\tau)_{\tau \in \text{Hom}(F, \Omega)}$ such that $\lambda_{\tau_{\infty, i}} = -\lambda_{\tau, d+1-i}$, where $c$ is the nontrivial element of $\text{Gal}(F^+/F)$.

If $F$ is a number field, $\chi : F^x \setminus A_F^x \rightarrow \mathbb{C}^\times$ is a continuous character whose restriction to the connected component of $(F \otimes \mathbb{R})^x$ is given by $x \mapsto \prod_{v \in \text{Hom}(F, \mathbb{C})} x_v^{\lambda_v}$ for some integers $\lambda_v$, and $\iota : \overline{Q_\mathbb{p}} \sim \mathbb{C}$ is an isomorphism, we let $\chi_\iota : G_F \rightarrow \overline{Q_\mathbb{p}}^\times$ be the continuous character given by

$$\chi_\iota(\text{Art}(F(x))) = \iota^{-1}(\chi(x) \prod_{\iota \in \text{Hom}(F, \mathbb{C})} x_x^{\lambda_\iota} \prod_{\sigma \in \text{Hom}(F, \overline{Q_\mathbb{p}})} x_x^{\lambda_\sigma}. \tag{2.1.1}$$

2.1. Automorphic Galois representations. Let $F$ be either a CM or totally real number field with maximal totally real subfield $F^+$. Let $c \in G_{F^+}$ be a choice of complex conjugation.

2.1.1. Following [BLGGT14, §2.1], we will say a pair $(\Pi, \chi)$ is a polarized automorphic representation of $\text{GL}_d(\mathbb{A}_F)$ if

- $\Pi$ is an automorphic representation of $\text{GL}_d(\mathbb{A}_F)$;
- $\chi : (F^+)^x \setminus A_F^x \rightarrow \mathbb{C}^\times$ is a continuous character such that for all $\nu|\infty$, the value $\chi_\nu(-1)$ is independent of $\nu$, and equals $(-1)^d$ if $F$ is CM;
- $\Pi^c \cong \Pi^c \otimes (\chi \circ \text{Nm}_{F^+/F} \circ \text{det})$.

We say that an automorphic representation $\Pi$ of $\text{GL}_d(\mathbb{A}_F)$ is polarizable if there is a character $\chi$ such that $(\Pi, \chi)$ is a polarized automorphic representation. If $F$ totally real field and $(\Pi, 1)$ is polarized, then we say that $\Pi$ is self-dual. If $F$ is CM and $(\Pi, \delta_\iota^{d/2}_{F^+/F^+})$ is polarized, then we say that $\Pi$ is conjugate self-dual. Recall that an automorphic representation $\Pi$ of $\text{GL}_d(\mathbb{A}_F)$ is called regular algebraic if $\Pi_{\infty}$ has the same infinitesimal character as an irreducible algebraic representation of $\text{Res}_{F/\mathbb{Q}} \text{GL}_d$. If $\lambda = (\lambda_\tau) \in (\mathbb{Z}_+^d)^{\text{Hom}(F, \mathbb{C})}$, then we let $\xi_\lambda$ denote the irreducible algebraic representation of $\text{Res}_{F/\mathbb{Q}} \text{GL}_d$ which is the tensor product over $\tau \in \text{Hom}(F, \mathbb{C})$ of the irreducible algebraic representations with highest weight $\lambda_\tau$. We say a regular algebraic automorphic representation $\Pi$ of $\text{GL}_d(\mathbb{A}_F)$ has weight $\lambda \in (\mathbb{Z}_+^d)^{\text{Hom}(F, \mathbb{C})}$ if $\Pi_{\infty}$ has the same infinitesimal character as $\xi_\lambda$. We will say a polarized automorphic representation $(\Pi, \chi)$ of $\text{GL}_d(\mathbb{A}_F)$ is cuspidal if $\Pi$ is. We will say a polarized automorphic representation $(\Pi, \chi)$ of $\text{GL}_d(\mathbb{A}_F)$ is regular algebraic if $\Pi$ is. In this case $\chi$ is necessarily an algebraic character.

We have the following theorem, due to the work of many people. We refer the reader to [BLGGT14, Theorem 2.1.1] and the references contained there (noting that the assumption of an Iwahori fixed vector in part (4) of [BLGGT14, Theorem 2.1.1] can be removed by the main result of [Car14]).

**Theorem 2.1.2.** Let $F$ be either a CM or totally real number field with maximal totally real subfield $F^+$. Let $(\Pi, \chi)$ be a regular algebraic, polarized, cuspidal automorphic representation of $\text{GL}_d(\mathbb{A}_F)$, of weight $\lambda \in (\mathbb{Z}_+^d)^{\text{Hom}(F, \mathbb{C})}$. Fix a rational prime $p$ and an isomorphism $\iota : \overline{Q_\mathbb{p}} \sim \mathbb{C}$. Then there is a continuous semisimple representation

$$\rho_{\Pi, \iota} : G_F \rightarrow \text{GL}_d(\overline{Q_\mathbb{p}})$$

satisfying the following properties.
1. There is a perfect symmetric pairing $\langle \cdot, \cdot \rangle$ on $\mathbb{Q}_p^d$ such that for any $a, b \in \mathbb{Q}_p^d$ and $\sigma \in G_F$,

$$\langle \rho_{\Pi,s}(\sigma)a, \rho_{\Pi,s}(\sigma)c \rangle = (\epsilon^{1-d}(\chi_\rho)(\sigma))\langle a, b \rangle.$$ 

2. For all $w|p$, $\rho_{\Pi,s}|_{G_w}$ is potentially semistable, and for any continuous embedding $\tau : F_w \hookrightarrow \overline{\mathbb{Q}}_p$,

$$\text{HT}_\tau(\rho_{\Pi,s}|_{G_w}) = \{\lambda, \tau + d - j \}_{j=1,\ldots,d}.$$ 

3. For any finite place $w$,

$$\iota\text{WD}(\rho_{\Pi,s}|_{G_w})^{F-sus} \cong \text{rec}_{F_w}(\Pi_w).$$

We note that an argument using the Baire category theorem and the compactness of $G_F$ shows that we can assume $\rho_{\Pi,s}$ takes values in $\text{GL}_d(\mathcal{O})$ with $\mathcal{O}$ the ring of integers of some finite extension of $\mathbb{Q}_p$, and that the perfect pairing $\langle \cdot, \cdot \rangle$ descends to a perfect pairing on $\mathcal{O}^d$.

### 2.2. Definite unitary groups.

In this subsection, we assume that $p$ is odd. We recall some constructions from [CHT08, §3.3] and [Gue11, §2] (see also [Tho12, §6]). Before doing so, we note that in [CHT08, §3.3] there are running assumptions that a certain ramification set denoted $S(B)$ there is nonempty, that (in our notation) $p > d$, and that $F$ is the composite of a quadratic imaginary field and $F^+$. None of these assumptions are necessary for what we need.

Let $F$ be a CM field with maximal totally real subfield $F^+$. Let $c$ denote the nontrivial element of $\text{Gal}(F/F^+)$. We assume that $F/F^+$ is unramified at all finite places and that every place above $p$ in $F^+$ splits in $F$. We assume that $F^+ \neq \mathbb{Q}$ and that if $d$ is even, then

$$d[F^+ : \mathbb{Q}] \equiv 0 \pmod{4}.$$ 

We also fix a finite extension $E$ of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $F$. We assume $E$ contains all embeddings of $F$ into an algebraic closure of $E$.

#### 2.2.1. Let $B = M_d(F)$, and let $g \mapsto g^*$ be an involution of the second kind on $B$. Then the pair $(B, \ast)$ defines a reductive $F^+$-group $G$ by

$$G(R) = \{ g \in B \otimes_{F^+} R \mid gg^* = 1 \}$$ 

for any $F^+$-algebra $R$. Since $d[F^+ : \mathbb{Q}] \equiv 0 \pmod{4}$ if $d$ is even, we can choose the involution $g \mapsto g^*$ on $B$ such that

(a) $G \otimes_{F^+} F_v^+$ is quasi-split for every finite place $v$ of $F^+$;

(b) $G(F_v^+) \cong U_d(\mathbb{R})$, the totally definite unitary group, for every infinite place $v$ of $F^+$.

Since $B = M_d(F)$, the data of $\ast$ is equivalent to a Hermitian form $h$ on $F^d$, and a choice of lattice $\mathcal{L}$ in $F^d$ such that $h(\mathcal{L} \times \mathcal{L}) \subseteq \mathcal{O}_F$ yields a maximal order $\mathcal{O}_B$ of $B$ such that $\mathcal{O}_B = \mathcal{O}_F$. This maximal order defines a model of $G$ over $\mathcal{O}_{F^+}$, that we again denote by $G$. For a finite place $v$ of $F^+$ that splits in $F$, we can find an isomorphism $\iota_v : \mathcal{O}_{B,v} \xrightarrow{\sim} M_d(\mathcal{O}_F) \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_{F_v^+}$ such that $\iota_v(g^*) = \iota_v(g)^c$. Writing $v = w w^c$, the choice of $w$ gives an isomorphism $\iota_w : G(\mathcal{O}_{F^+}) \cong \text{GL}_d(\mathcal{O}_{F_w})$ by $\iota_w^{-1}(g, (g^c)^{-1}) \mapsto g$. This extends to an isomorphism $\iota_w : G(F_v^+) \cong \text{GL}_d(F_w)$, and $\iota_{w^c} = \iota((c \circ \iota_w)^{-1}).$

Let $S_p$ denote the set of places of $F^+$ above $p$. For each $v \in S_p$, fix a choice $\tilde{v}$ of place of $F$ dividing $v$, and let $\tilde{S}_p = \{ \tilde{v} \mid v \in S_p \}$. Let $J_p$ be the set of embeddings $F^+ \hookrightarrow E$, and let $\tilde{J}_p$ be the set of embeddings $F \hookrightarrow E$ that give rise to a place in $\tilde{S}_p$. Thus, restriction to $F^+$ gives a bijection $\tilde{J}_p \cong J_p$. Given $\lambda \in \mathbb{Z}_+^d$, we let

$$\xi_\lambda : \text{GL}_d \rightarrow \text{GL}(W_\lambda)$$

denote the irreducible algebraic representation defined over $\mathbb{Q}$ with highest weight

$$\text{diag}(t_1, \ldots, t_d) \rightarrow \prod_{i=1}^d t_i^{\lambda_i}.$$
We choose a $GL_d(O)$-stable lattice $M_\lambda$ in $W_\lambda \otimes_{Q} E$. Then for any $\lambda = (\lambda_\tau) \in (Z^+_P)^0_{\Hom(F,E)}$, we define a representation
\[
\xi_\lambda : G(F^+_p) \longrightarrow GL(\bigotimes_{\tau \in J_p} W_{\lambda_\tau})
\]
\[
g \longmapsto \bigotimes_{\tau \in J_p} \xi_{\lambda_\tau}(\tau g),
\]
where we have written $\iota_\tau$ for $\iota_w$ if $\tau$ gives rise to the place $w$. The $O$-lattice $M_\lambda := \bigotimes_{\tau \in J_p} M_{\lambda_\tau}$ in the $E$-vector space $W_\lambda := \bigotimes_{\tau \in J_p} W_{\lambda_\tau}$ is stable under $G(O_{F^+_p})$.

If $A$ is any $O$-algebra and $U$ is an open compact subgroup of $G(A_{F^+_p}^\infty)$ such that $U_p \subseteq G(O_{F^+_p})$, then we define a space of automorphic forms $S_\lambda(U,A)$ to be the space of functions
\[
f : G(F^+_p) \backslash (A_{F^+_p}^\infty) \longrightarrow M_\lambda \otimes_O A
\]
such that $f(gu) = u_p^{-1}f(g)$ for all $g \in G(A_{F^+_p}^\infty)$ and $u \in U$. If $V$ is any compact subgroup of $G(A_{F^+_p}^\infty)$ such that $V \subseteq G(O_{F^+_p})$, then we define
\[
S_\lambda(V,A) = \lim_{V \subseteq U} S_\lambda(U,A),
\]
with the limit over all open compact subgroups $U$ containing $V$ such that $U_p \subseteq G(O_{F^+_p})$. Note that if $A$ is flat over $O$, then $S_\lambda(V,A) = S_\lambda(V,O) \otimes_O A$.

**Proposition 2.2.2.** Fix an embedding $\iota : E \hookrightarrow C$, and view $C$ as an $O$-algebra via $\iota$.

1. $S_\lambda(\{1\},C)$ is a semisimple admissible representation of $G(A_{F^+_p}^\infty)$.
2. Let $\Pi$ be a regular algebraic conjugate self-dual representation of $GL_d(A_{F^+_p})$ of weight $\iota \lambda$. Then there is an irreducible subrepresentation $\pi = \otimes_v \pi_v$ of $S_\lambda(\{1\},C)$ such that the following hold.
   - If $v$ is a finite place of $F^+$ that splits as $v = w w^c$ in $F$, then $\pi_v \cong \Pi_v \circ \iota_w$.
   - If $v$ is a finite place of $F^+$ inert in $F$, and $\Pi_v$ is unramified, then $\pi_v$ has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^+)$.

**Proof.** For part 1., see [CHT08, part 1 of Proposition 3.3.2]. For part 2., [Lab, Théorème 5.4] implies that there is an irreducible subrepresentation $\pi = \otimes_v \pi_v$ of $S_\lambda(\{1\},C)$ such that $\Pi$ is a weak base change (see [Lab, §4.10]) of $\pi$. Then applying [Lab, Corollaire 5.3] and strong multiplicity one for $GL_d$ gives the result. \qed

2.2.3. Now fix an open compact subgroup $U$ of $G(A_{F^+_p}^\infty)$ such that $U_p \subseteq G(O_{F^+_p})$. Fix a set of finite places $S$ of $F^+$ containing $S_p$ and all finite places at which $U$ is not hyperspecial maximal compact. For each finite place $w$ of $F$ split over $v \notin S$ of $F^+$, the Hecke operators
\[
T^{(j)}_w = \iota_w^{-1} \left[ \left[ GL_d(O_{F_w}) \left( \begin{array}{cc} \varpi_w^{-1} & 1 \\ 0 & 1 \end{array} \right) GL_d(O_{F_w}) \right] \right]
\]
for $j \in \{1, \ldots, d\}$, as well as $(T^{(d)}_w)^{-1}$, act on $S_\lambda(U,A)$. Here $\varpi_w$ denotes a choice of uniformizer of $F_w$, and the operators $T^{(j)}_w$ do not depend in the choice of $\varpi_w$. We let $T^S_\lambda(U,A)$ be the $A$-subalgebra of $\End_A(S_\lambda(U,A))$ generated by the $T^{(1)}_w, \ldots, T^{(d)}_w, (T^{(d)}_w)^{-1}$, for all $w$ as above. If $A = O$, then we write $T^S_\lambda(U)$ for $T^S_\lambda(U,O)$.

**Corollary 2.2.4.** Let $L/E$ be finite extension, and $x : \mathbb{T}_\lambda^S(U)[1/p] \rightarrow L$ be an $E$-algebra morphism. There is a continuous absolutely irreducible representation
\[
\rho_x : Gal(F(S)/F) \longrightarrow GL_d(L)
\]
such that the following hold.

1. There is a perfect symmetric paring $\langle , \rangle$ on $L^d$ such that for any $a,b \in L^d$ and $\sigma \in G_F$,
\[
\langle \rho_x(\sigma)(a), \rho_x(\sigma)(b) \rangle = \epsilon^{1-d}(\sigma) \langle a, b \rangle.
\]
2. For any finite place $w$ in $F$ split over $v \notin S$ in $F^+$, the characteristic polynomial of $\rho_x(\text{Frob}_w)$ is
\[
X^d + \cdots + (-1)^j \text{Nm}(w) \frac{d-1}{2} x(T^{(j)}_w) X^{d-j} + \cdots + (-1)^d \text{Nm}(w) \frac{d(d-1)}{2} x(T^{(d)}_w).
\]
Lemma 2.2.5. Let $\rho_x|_{G_w}$ be potentially semistable, and is semistable if $\iota_w(U_v)$ contains the Iwahori subgroup, where $v$ denotes the place of $F^+$ below $w$. For any $\tau \in J_p$,
$$HT_\tau(\rho_x|_{G_w}) = \{\lambda_{\tau,j} + d - j\}_{j=1,\ldots,d}.$$ 

In particular, $\rho_x|_{G_w}$ has regular $p$-adic Hodge type.

Proof. The finite flat $O$-algebra $T^S_\lambda(U)$ acts faithfully on
$$S_\lambda(U) \otimes_O \bar{\mathbb{Q}}_p \cong S_\lambda(U, \bar{\mathbb{Q}}_p) = \bigoplus_\pi \pi^U,$$
with the direct sum taken over irreducible subrepresentations of $S_\lambda(\{1\}, \bar{\mathbb{Q}}_p)$, and $T\lambda(U)$ acts on each $\pi^U$ by some $E$-algebra morphism $x_\pi : T\lambda(U)[1/p] \to \bar{\mathbb{Q}}_p$. Thus, $x \otimes L \bar{\mathbb{Q}}_p = x_\pi$ for some irreducible subrepresentation $\pi$ of $S_\lambda(\{1\}, \bar{\mathbb{Q}}_p)$. We now apply [Gue11, Theorem 2.3], letting $\rho_x$ be the representation denoted $r_p(\pi)$ in [Gue11, Theorem 2.3], noting that:

- We may assume that $\rho_x$ takes values in $L$ by Chebotarev density, since $x_\pi(T_w^{(j)}) = x(T_w^{(j)}) \in L$ for all $w$ of $F$ split over $v \notin S$ in $F^+$ and $j \in \{1, \ldots, d\}$.

- We may assume that the pairing is symmetric by applying [BC11, Theorem 1.2] to the construction in the proof of [Gue11, Theorem 2.3]. More specifically, the construction in [Gue11, Theorem 2.3] shows there is a partition $d = d_1 + \cdots + d_r$ and factorizations $d_i = a_i b_i$, such that
$$\rho_x = \bigoplus_{i=1}^r g_i \otimes \eta_i \otimes (1 \oplus \epsilon \oplus \cdots \oplus \epsilon^{b_i-1}),$$
where $g_i$ is the Galois representation associated to a regular algebraic conjugate self-dual cuspidal automorphic representation of $GL_{a_i}(\mathbb{A}_F)$, and $\eta$ is a character satisfying $\eta \epsilon^c = \epsilon^{1+a_i-b_i-d}$. There is a symmetric perfect pairing for $g_i$ with multiplier $\epsilon^{1-a_i}$ by part 1. of 2.1.2 (which uses [BC11, Theorem 1.2]). This defines a symmetric perfect pairing for $g_i \otimes \eta$ with multiplier $\epsilon^{2-b_i-d}$. Defining the obvious symmetric pairing on $(1 \oplus \cdots \oplus \epsilon^{b_i-1})$ with multiplier $\epsilon^{b_i-1}$, tensoring these two, and taking the direct sum over $i$ yields a symmetric perfect pairing for $\rho_x$ with multiplier $\epsilon^{1-d}$.

Recall the group scheme $G_d$ and its canonical character $\nu : G_d \to GL_1$ of §1.3. The following is a standard application of 2.2.4 and 1.3.1.

Lemma 2.2.5. Let $m$ be a maximal ideal of $T^S_\lambda(U)$. There is a continuous homomorphism
$$\tau_m : \text{Gal}(F(S)/F^+) \longrightarrow G_d(T^S_\lambda(U)/m)$$
with $\tau_m^{-1}(G_d^0(T^S_\lambda(U)/m)) = \text{Gal}(F(S)/F)$, satisfying the following:

1. If $w$ is a finite place of $F$ split over $v \notin S$ of $F^+$, then $\tau_m(\text{Frob}_w)$ has characteristic polynomial
$$X^d + \cdots + (-1)^{d-1} \text{Nm}(w)^{\frac{(d-1)(d-2)}{2}} T_w^{(j)} X^{d-j} + \cdots + (-1)^d \text{Nm}(w)^{\frac{d(d-1)}{2}} T_w^{(d)} \mod m.$$

2. $\nu \circ \tau_m = \epsilon^{1-d} \delta_{F/F^+} \mod m_O$.

We now fix a maximal ideal $m$ of $T^S_\lambda(U)$, and let
$$\tau_m : \text{Gal}(F(S)/F^+) \longrightarrow G_d(T^S_\lambda(U)/m)$$
be as in 2.2.5. Enlarging $E$ if necessary, we may assume that $T^S_\lambda(U)/m = F$. Arguing exactly as in the proof of [CHT08, Proposition 3.4.4], we have the following lemma.

Lemma 2.2.6. Assume that $\tau_m|_{G_F}$ is absolutely irreducible. Then there is a continuous lift
$$\tau_m : \text{Gal}(F(S)/F^+) \longrightarrow G_d(T^S_\lambda(U)/m)$$
of $\tau_m$, unique up to conjugacy, satisfying the following:

1. If $w$ is a finite place of $F$ split over $v \notin S$ of $F^+$, then $\tau_m(\text{Frob}_w)$ has characteristic polynomial
$$X^d + \cdots + (-1)^d \text{Nm}(w)^{\frac{d(d-1)}{2}} T_w^{(j)} X^{d-j} + \cdots + (-1)^d \text{Nm}(w)^{\frac{d(d-1)}{2}} T_w^{(d)}.$$ 

2. $\nu \circ \tau_m = \epsilon^{1-d} \delta_{F/F^+}$.
Lemma 2.2.7. Assume that \( \varpi \in \mathcal{O} \) contains the Iwahori subgroup for each \( \nu \in \mathcal{S}_p \).

1. The lift \( \rho_{\varphi} : \text{Gal}(F(S)/F^+) \rightarrow \mathcal{G}_d(T^\lambda_S(U)_m) \) of \( \varphi \) from 2.2.6 is of type \( S \), and the induced CNL\( \mathcal{O} \)-morphism \( R_S \rightarrow T^\lambda_S(U)_m \) is surjective.

2. Denote again by \( x \) the composite \( R_S[1/p] \rightarrow T^\lambda_S(U)_m[1/p] \xrightarrow{\rho_{\varphi}} E \). The localization and completion \( (R_S)_{\lambda}^\lambda \) acts freely on \( S_{\lambda}(U)_{\lambda}^\lambda \) if and only if \( (R_S)_{\lambda}^\lambda = E \).

Proof. The fact that \( r_m \) factors through \( \text{Gal}(F(S)/F^+) \) and has \( \nu \circ r_m = \epsilon_{G,F}^+ \rho_{\varphi}[1/p] \) is contained in 2.2.6. To show that \( r_m \) is of type \( S \), it remains to show that for every \( \nu \in S \), the local lift \( r_m|_{\mathcal{O}_\nu} \) factors through the given local lifting ring \( R_w \). For \( w \in \tilde{S} \setminus \tilde{S}_p \), this is automatic, so we only need to show it for \( \tilde{v} \in \tilde{S}_p \).

Since \( T^\lambda_S(U)_m \) is reduced and finite flat over \( \mathcal{O} \), it suffices to show that for any finite extension \( L/E \), and any \( E \)-algebra morphism \( y : T^\lambda_S(U)_m[1/p] \rightarrow L \), that the representation

\[
\rho_y : \text{Gal}(F(S)/F) \xrightarrow{r_m|_{\mathcal{O}_\nu}} \text{GL}_d(T^\lambda_S(U)_m) \xrightarrow{\rho_{\varphi}} \text{GL}_d(L)
\]

induces an \( E \)-algebra morphism \( R_{\varphi}^\lambda[1/p] \rightarrow L \) that factors through \( R_{\varphi}^\lambda[1/p] \rightarrow T^\lambda_S(U)_m[1/p] \) for every \( \tilde{\nu} \in \tilde{S}_p \). This happens if and only if for every \( \tilde{\nu} \in \tilde{S}_p \), the local representation \( \rho_{\varphi}|_{\mathcal{O}_{\nu}} \) is semistable with \( \rho_{\varphi}[1/p] \) of type \( S \). This follows from part 3. of 2.2.4, since \( \varpi(U_i) \) contains the Iwahori subgroup for every \( \nu \in \mathcal{S}_p \), and the \( \rho_{\varphi}[1/p] \) of type \( S \) depends only on its associated graded, which in turns depends only on \( \lambda \). This shows that \( r_m \) is of type \( S \).

The induced CNL\( \mathcal{O} \)-algebra map is surjective follows from part 1. of 2.2.6 and Chebotarev density. For part 2., note that \( S_{\lambda}(U)^\lambda = S_{\lambda}(U) \) is a product of fields corresponding to eigenforms for \( T^\lambda_S(U) \) in the same \( T^\lambda_S(U) \)-eigensclass, so \( (R_S)_{\lambda}^\lambda \) acts diagonally on this product of fields, which implies it acts through its surjection to its residue field \( E \).

3. The main theorems

Throughout this section \( p \) will be an odd prime and \( E \) will be a finite extension of \( \mathbb{Q}_p \) with ring of integers \( \mathcal{O} \) and residue field \( \mathbb{F} \). Let \( F \) be a CM field with maximal totally real subfield \( F^+ \). Let \( \mathcal{C} \in G_{F^+} \) be a fixed choice of complex conjugation. Let \( \mathcal{S} \) be a finite set of places of \( F^+ \) containing all places above \( p \).

3.1. The main theorem. In this subsection we prove our main theorem on adjoint Selmer groups. Before doing so, we first recall the definition of an adequate subgroup of \( \text{GL}_d(F) \) and of \( \text{GL}_d(F) \) by Thorne, and the relation between the two.

Let \( \Gamma \) be a subgroup of \( \text{GL}_d(F) \), and assume that \( \mathbb{F} \) contain all eigenvalues of all elements of \( \Gamma \). Let \( \mathfrak{g}_d = \mathfrak{g}_d(F) \) be the Lie algebra of \( \text{GL}_d(F) \) with the adjoint \( \Gamma \)-action. Let \( \mathfrak{h} \subset \mathfrak{g}_d \) be the centre of \( \mathfrak{g}_d \); note \( \mathfrak{h} \cong \mathbb{F} \). Identifying \( \mathfrak{g}_d \) with \( \text{End}_d(F^d) \), for any \( g \in \Gamma \) and eigenvalue \( \alpha \) of \( g \), we let \( e_{g,\alpha} \in \mathfrak{g}_d \) be the unique \( \mathfrak{g} \)-equivariant projection of \( F^d \) onto the generalized \( \alpha \)-eigenspace of \( g \). The following is [GHT14, Definition A.1].

**Definition 3.1.1.** We say \( \Gamma \) is adequate if the following hold:

1. \( H^1(\Gamma, \mathfrak{g}_d/\mathfrak{h}) = 0 \) for \( i = 0, 1 \);
2. \( H^1(\Gamma, \mathfrak{g}_d) = 0 \);
3. for every irreducible \( \mathbb{F}[\Gamma] \)-submodule \( W \) of \( \mathfrak{g}_d \), there is \( g \in \Gamma \) and an eigenvalue \( \alpha \) of \( g \) such that \( \text{tr}(e_{g,\alpha}W) \neq 0 \).

We note that this is slightly more general than the definition of adequate found in much of the literature (i.e. [Tho12, Definition 2.3]). In particular, it allows \( p \mid d \).

Now let \( \mathcal{G} \) be a subgroup of \( \text{GL}_d(F) \), and let \( \mathcal{G}^d = \mathcal{G} \cap \mathcal{G}_d(F) \). The following is [Tho12, Definition 2.3] (see also [GHT14, Definition A.2]).
Definition 3.1.2. We say that $G$ is adequate if the following hold:

1. $H^i(G, \mathfrak{g}_d) = 0$ for $i = 0, 1$;
2. $H^1(G, \mathbb{F}) = 0$;
3. for every irreducible $\mathbb{F}[G]$-submodule $W$ of $\mathfrak{g}_d$, there is $g \in G^0$ and an eigenvalue $\alpha$ of $g$ such that $\text{tr}(\epsilon g, \alpha W) \neq 0$.

We have the following lemma of Thorne [GHT14, Lemma A.3].

Lemma 3.1.3. Let $G$ be a subgroup of $\mathcal{G}_d(\mathbb{F})$ that surjects onto $\mathcal{G}_d(\mathbb{F})/\mathcal{G}_d^0(\mathbb{F})$, and let $\Gamma$ be the image of $G \cap \mathcal{G}_d^0(\mathbb{F})$ under the projection $\mathcal{G}_d^0(\mathbb{F}) \to \mathcal{G}_d(\mathbb{F})$. If $\Gamma$ is adequate in the sense of 3.1.1, then $G$ is adequate in the sense of 3.1.2.

3.1.4. Let $r : \text{Gal}(F(S)/F^+) \to \mathcal{G}_d(E)$ be a continuous homomorphism. We recall that $\text{ad}(r)$ denotes $\mathfrak{g}_d(E)$ with the adjoint Gal($F(S)/F^+$)-action $\text{ad} \circ r$. After conjugating, we can assume that $r$ takes values in $\mathcal{G}_d(\mathcal{O})$, and we let $\mathfrak{r} = r \otimes \mathcal{O} \mathbb{F}$.

Theorem 3.1.5. Assume there is a finite extension $L/F$ of CM fields, a regular algebraic polarizable cuspidal automorphic representation $\Pi$ of $\text{GL}_d(\mathbb{A}_L)$, and an isomorphism $\iota : \overline{\mathcal{O}}_p \cong \mathbb{C}$ such that the following hold:

(a) $r|_{\mathcal{G}_L} \otimes \overline{\mathcal{O}}_p \cong \rho_{L, i}$;
(b) $\zeta_p \notin L$ and $\mathfrak{r}(G_{L(L_p)})$ is adequate.

Then the following hold.

1. $H^2_\mathfrak{r}(F(S)/F^+, \text{ad}(r)) = 0$.
2. $H^2(F(S)/F^+, \text{ad}(r)) = 0$.
3. The natural map

$$H^1(F(S)/F^+, \text{ad}(r)) \to \prod_{\nu|p} H^1(F_{\nu}^+, \text{ad}(r))/H^1_\mathfrak{r}(F_{\nu}^+, \text{ad}(r))$$

is an isomorphism.

Proof. Using 1.3.5, parts 2. and 3. are implied by 1. To prove $H^1_\mathfrak{r}(F(S)/F^+, \text{ad}(r)) = 0$, we first note that we are free to enlarge $S$, so we can (and do) assume that $S$ contains at least one finite place not above $p$. We are also free to replace $F$ with any finite extension $L/F$ of CM fields. Indeed, let $L^+$ denote the maximal totally real subfield of $L$ and $S_L^+$ the set of places of $L^+$ above $S$. Then $F(S)L^+$ is Galois over $L^+$ and $\text{Gal}(F(S)L^+/L^+)$ canonically isomorphic to an open subgroup $H$ of $\text{Gal}(F(S)/F^+)$. The restriction on cohomology to $H$ is injective by considering restriction-corestriction, then via the isomorphism $H \cong \text{Gal}(F(S)L^+/L^+)$ and inflation to $\text{Gal}(L(S_L^+)/L^+)$, we have an injection

$$H^1(F(S)/F^+, \text{ad}(r)) \to H^1(L(S_L^+)/L^+, \text{ad}(r))$$

such that the following diagram commutes

$$\begin{array}{ccc}
H^1(F(S)/F^+, \text{ad}(\rho)) & \to & \prod_{\nu \in S} H^1(F_{\nu}^+, B_{\text{dr}} \otimes \mathbb{Q}_p, \text{ad}(\rho)) \\
\downarrow & & \downarrow \\
H^1(L(S_L^+)/L^+, \text{ad}(\rho)) & \to & \prod_{\nu \in S_L^+} H^1(L_{\nu}^+, B_{\text{dr}} \otimes \mathbb{Q}_p, \text{ad}(\rho)).
\end{array}$$

Thus $H^1_\mathfrak{r}(F(S)/F^+, \text{ad}(r))$ injects into $H^1(L(S_L^+)/L^+, \text{ad}(r))$, so the former is trivial if the latter is. In particular, we may assume $L = F$ in the statement of the theorem.

Note that if $\chi : \text{Gal}(F(S)/F) \to \mathcal{O}^\times$ is any continuous character, then by 1.3.1 there is a continuous homomorphism $r \otimes \chi : \text{Gal}(F(S)/F^+) \to \mathcal{G}_d(\mathcal{O})$ such that $(r \otimes \chi)|_{\mathcal{G}_p} = r|_{\mathcal{G}_p} \otimes \chi$ and $\nu \circ (r \otimes \chi) = (\chi \nu \nu)$. There is an isomorphism $\text{ad}(r) \cong \text{ad}(r \otimes \chi)$ of $\text{Gal}(F(S)/F^+)$-modules, so using [CHT08, Lemma 4.1.4], we can twist $r$ and $\Pi$, and assume $\Pi$ is conjugate self-dual.

Let $S_p$ be the set of places above $p$ in $F^+$, and for each $\nu \in S_p$, choose some $\tilde{\nu} | \nu$ in $F$. Let $\overline{S}_p = \{ \tilde{\nu} | \nu \in S_p \}$. Using cyclic base change [AC89, Theorem 4.2 of Chapter 3], further replacing $F$ by a solvable extension, we may assume that $F/F^+$, $r$, $\mathfrak{r}$, and $\Pi$ satisfy:
(i) $F/F^+$ is unramified at all finite places, and every $v \in S$ splits in $F$;
(ii) if $d$ is even, then $d[F^+:Q] \equiv 0 \pmod{4}$;
(iii) $\pi|_{G_F} \otimes \mathbb{Q}_p \cong \rho_{H^+}$, and $\Pi$ is conjugate self-dual;
(iv) $\zeta_p \not\in F$ and $\pi(G_{F(\zeta_p)})$ is adequate;
(v) $\Pi_\varepsilon$ has Iwahori fixed vectors for every $\tilde{v} \in \tilde{S}_p$.

Enlarging $E$ if necessary, we assume that $E$ contains all embedding of $F$ into $\mathbb{Q}_p$, and that $F$ contains the eigenvalues of all elements in the image of $\pi|_{G_F}$. Let $\lambda \in (\mathbb{Z}_d)^{\operatorname{Hom}(F,E)}$ be such that $\lambda$ is the weight of $\Pi$. Let $G$ be the $\mathcal{O}_{F^+}$-group scheme of 2.2.1. Recall that for any finite place $w$ of $F$ split over $F^+$, we have an isomorphism $\iota_w : G_{F^+} \cong \mathbb{G}_m$. For any $\mathcal{O}_d$-algebra $A$, and compact subgroup $V$ of $G(\mathbb{A}_{F^+}^\infty)$, we let $S\chi(V,A)$ be the $A$-module of automorphic forms of 2.2.1. Viewing $\mathbb{C}$ as an $\mathcal{O}_d$-algebra via $\iota$, by 2.2.2 there is an irreducible subrepresentation $\pi$ of $S\chi((1),\mathbb{C})$ such that

(vi) if $v$ is a finite place of $F^+$ that splits as $v = \wp \mathbb{C}$ in $F$, then $\pi_v \cong \Pi_v \circ \iota_w$;
(vii) if $v$ is a finite place of $F^+$ inert in $F$, and $\Pi_v$ is unramified, then $\pi_v$ has a fixed vector for some hyperspecial maximal compact subgroup of $G(F_v^+)$. 

Now choose an open compact subgroup $U \subseteq G(\mathbb{A}_{F^+}^\infty)$ such that $U_p \subseteq G(\mathcal{O}_{F^+})$ and satisfying the following:

(viii) $\iota \iota(U_v)$ is the Iwahori subgroup of $\mathbb{G}_m$ for each $v \in S_p$;
(ix) $\pi \mathbb{C} \neq 0$;
(x) $U_v$ is a hyperspecial maximal compact subgroup for every $v \in S$;
(xi) for all $t \in G(\mathbb{A}_{F^+})$, the group $t^{-1}(G(F^+) \cap U)$ contains no element of order $p$.

Since we have assumed that $S$ contains at least one finite place not above $p$, assumptions (viii), (x), and (xi) can be satisfied simultaneously by letting $U_v$ be sufficiently small for some $u \in S \setminus S_p$. Let $T\chi^+(U)$ be the Hecke algebra in 2.2.3. Since $S\chi(U_\mathbb{C}) \cong S\chi(U_\mathbb{C}) \otimes_{\mathbb{Q}_p} \mathbb{C}$, there is an action of $T\chi^+(U)$ on $\pi \mathbb{C}$ and it acts via an $E$-algebra morphism $x : T\chi^+(U) \to E$. Indeed, $T\chi^+(U)$ acts on $\pi \mathbb{C}$ via a homomorphism $x : T\chi^+(U)[1/p] \to E$ that factors through $E$ by assumptions (iii) and (vii), local-global compatibility (i.e. part 3. of 2.1.2), and that $r|_{G_F}$ takes values in $\mathbb{GL}_d(E)$. Moreover, the Galois representation $\rho_x : \mathbb{Gal}(F(S)/F^+) \to \mathbb{GL}_d(E)$ of 2.2.4 is isomorphic to $r|_{G_F}$.

Let $\mathfrak{m}$ be the maximal ideal of $T\chi(U)$ containing $\ker(x)$. We can choose a continuous homomorphism

$$\tau : \mathbb{Gal}(F(S)/F^+) \to \mathbb{GL}_d(T\chi(U))/\mathfrak{m} = \mathbb{GL}_d(F)$$

as in 2.2.5, such that $\tau_\mathfrak{m} = \tau$. In particular, $\tau_\mathfrak{m}|_{G_F}$ is absolutely irreducible. Then 2.2.6 gives a lift

$$r_\mathfrak{m} : \mathbb{Gal}(F(S)/F^+) \to \mathbb{GL}_d(T\chi(U))/\mathfrak{m}$$

of $\tau_\mathfrak{m}$, and letting $r_x$ be the composite

$$\mathbb{Gal}(F(S)/F^+) \xrightarrow{r_\mathfrak{m}} \mathbb{GL}_d(T\chi(U))/\mathfrak{m} \xrightarrow{\pi} \mathbb{GL}_d(E),$$

define $r_x$. The lifts $r_x$ and $r$ of $\tau$ define the same deformation.

For each $v \in S \setminus S_p$, let $\tilde{v}$ be a choice of place in $F$ dividing $v$, and let $\tilde{S} = \{ \tilde{v} \mid v \in S \}$. For each $\tilde{v} \in \tilde{S}_p$, let $\mathfrak{v}_\tilde{v} = D_{\mathfrak{Hv}}(\rho|_{G_{\mathfrak{v}}})$, the $p$-adic Hodge type of $\rho$ at $\tilde{v}$. We consider the global deformation datum (see 1.3.7)

$$S = (F/F^+, S, \tilde{S}, \mathcal{O}, \mathfrak{v}, \iota^1dG_{F^+}, \{ R_{\mathfrak{v}} \}_{\mathfrak{v} \in \tilde{S}_p})$$

where

- $R_{\mathfrak{v}} = R_{\mathfrak{v}}^\infty(1, \mathfrak{v}_\mathfrak{v})$ if $\mathfrak{v} \in \mathcal{C}_p$ (see 2.1.2) and
- $R_{\mathfrak{v}} = R_{\mathfrak{v}}^\infty$ if $\mathfrak{v} \in \tilde{S}_p \setminus \mathcal{C}_p$.

We let $R_S$ be the universal type $S$ deformation ring (see 1.3.8 and 1.3.9), and let $R_S^{\Sigma} = \hat{\mathbb{Z}}_{\tilde{S}_p}R_{\mathfrak{v}}$. By part 1. of 2.2.7, the deformation $[r_\mathfrak{m}]$ of $\tau_\mathfrak{m}$ is of type $S$, and there is a surjective complex $\mathcal{C}_p$-algebra morphism $R_S \to T\chi(U)_m$. Again denote by $x$ the $E$-algebra morphism $R_S[1/p] \to T\chi(U)_m[1/p] \xrightarrow{x} E$. Make a choice of lift for the universal type $S$ deformation so that the specialization $r_x$ of this lift via $x : R_S[1/p] \to E$ is equal to $r$. Then, by 1.3.12,

$$H^1_S(F(S)/F^+, \text{ad}(r)) = 0$$

if and only if $(R_S)_x^\wedge = E$, and this happens if and only if $(R_S)_x^\wedge$ acts faithfully on $S\chi(U)_x^\wedge$ by part 2. of 2.2.7.
Our choice of lift for the universal type $S$ deformation gives a CNL$_O$-morphism $R^{\text{ge}}_S \to R_S$. We denote again by $x$ the induced CNL$_O$-morphisms

\[ x : R^{\text{ge}}_S[1/p] \to R_S[1/p] \xrightarrow{\sim} E \quad \text{and} \quad x : R_S[1/p] \to R_S[1/p] \xrightarrow{\sim} E \quad \text{for each} \ \tilde{v} \in \tilde{S}. \]

Note that for each $\tilde{v} \in \tilde{S}$, the morphism $x : R_S[1/p] \to E$ corresponds to the local representation $r|_{G_{\tilde{v}}}$. Since $\Pi$ is a cuspidal representation of $\text{GL}_d(A_F)$, it is generic, and its local factors $\Pi_w$ at all finite places are generic. Then 1.1.3 implies that $\text{WD}(r|_{G_{\tilde{v}}}) \cong \iota^{-1}\text{rec}_{F_v}^T(\Pi_w)$ is generic for all finite places $w$. Then, by 1.2.2, 1.2.4, and 1.2.7,

\[- \text{if } \tilde{v} \in \tilde{S} \setminus \tilde{S}_p, \text{then} \ (R_{S})^{\text{ge}}_x \text{ is formally smooth over } E \text{ of dimension } d^2;\]
\[- \text{if } \tilde{v} \in \tilde{S}_p, \text{ then} \ (R_{S})^{\text{ge}}_x \text{ is formally smooth over } E \text{ of dimension } d^2 + \frac{d(d-1)}{2}[F_v : \mathbb{Q}] \text{ (here we used that the } p\text{-adic Hodge type of } r|_{G_{\tilde{v}}} = r_x|_{G_{\tilde{v}}} \text{ is regular by part 3.2.4}).\]

Then proof of [Kis09a, Lemma 3.4.12] shows

\[(\text{xii}) \ (R_{S})^{\text{ge}}_x \text{ is formally smooth over } E \text{ of dimension } d^2|S| + \frac{d(d-1)}{2}[F^+ : \mathbb{Q}].\]

We now use the argument of [Tho12, Theorem 6.8]. There are assumptions there that the local deformation rings at places $\tilde{v} \in \tilde{S}_p$ are crystalline deformation rings, and that $U_v = G(O_{F_v})$ for $v \in S_p$, but these assumptions are not used in what we need here. Since $\mathcal{O}_p \not\in F$ and $\mathcal{O}(G_{\tilde{v}})$ is adequate (in the sense of 3.1.2), the proof of [Tho12, Theorem 6.8], shows there are nonnegative integers $g$ and $q$, such that letting

\[- S_\infty = \mathcal{O}[[Z_1, \ldots, Z_d|S], Y_1, \ldots, Y_g]];\]
\[- a = (Z_1, \ldots, Z_d|S], Y_1, \ldots, Y_g) \subseteq S_\infty;\]
\[- R_\infty = R^{\text{loc}}_S[[X_1, X_2]]];\]

there is an $R_\infty$-module $M_\infty$ with a commuting action of $S_\infty$ such that the following hold.

\[(\text{xiii}) \text{ There is a local } \mathcal{O}\text{-algebra morphism } S_\infty \to R_\infty \text{ and a surjection } R_\infty \to R_S \text{ of } R^{\text{ge}}_S\text{-algebras with } aR_\infty \text{ in its kernel.}\]
\[(\text{xiv}) \text{ There is a surjection } M_\infty \to S_\lambda(U)_m \text{ of } R_\infty \text{ modules with kernel } aM_\infty, \text{ where the } R_\infty\text{-module structure of } S_\lambda(U)_m \text{ is via the surjection } R_\infty \to R_S \text{ of } (\text{xiii}).\]
\[(\text{xv}) \text{ } M_\infty \text{ is a finite free } S_\infty\text{-module.}\]
\[(\text{xvi}) \ g = q - \frac{d(d-1)}{2}[F^+ : \mathbb{Q}].\]

We note that [Tho12, Theorem 6.8] does not state (xiii), but only that the action of $S_\infty$ on $M_\infty$ (denoted $H_\infty$ there) factors through the action of $R_\infty$ on $M_\infty$. However, the patching patching part of the argument in [Tho12, Theorem 6.8] is quoted in [Tho12, Lemma 6.10] from [BLGG11, Theorem 3.6.1], where (xiii) is shown (in particular, see [BLGG11, Sublemma in the proof of Theorem 3.6.1]). Also, in (xvi) we have used that the $\mu_m$ in [Tho12, Theorem 6.8] is $d$ in our case.

Denote again by $x$ the $E$-algebra morphism $x : R_S[1/p] \to R_S[1/p] \xrightarrow{\sim} E$ and let $p_\infty = \ker(x) \in \text{Spec } R_\infty$. Note that $aR_\infty \subseteq p_\infty$. This together with the fact that $M_\infty$ is finite free over $S_\infty$ implies we can find an $M_\infty$-regular sequence of length $d^2|S| + q$ in $p_\infty$, and

\[\text{depth}_{(M_\infty)}(M_\infty)^{\lambda}_x \geq \text{depth}_{(M_\infty)}(p_\infty, M_\infty) \geq d^2|S| + q.\]

On the other hand, using (xii), (xiii), and (xvi) above, we deduce that $(R_{S})^{\text{ge}}_x \cong (R^{\text{loc}}_S)^{\lambda}_x[[X_1, \ldots, X_g]]$ is formally smooth over $E$ of dimension

\[d^2|S| + \frac{d(d-1)}{2}[F^+ : \mathbb{Q}] + g = d^2|S| + q.

The Auslander–Buchsbaum formula then implies that $(M_\infty)^{\lambda}_x$ is a finite free $(R_{S})^{\text{ge}}_x$-module. Then $(M_\infty)^{\lambda}_x/a$ is a finite free $(R_{S})^{\text{ge}}_x/a$-module. But $(M_\infty)^{\lambda}_x/a \cong S_\lambda(U)^{\lambda}_x$ and the action of $(R_{S})^{\text{ge}}_x/a$ on it factors through $(R_{S})^{\text{ge}}_x$. This shows $(R_{S})^{\text{ge}}_x$ acts faithfully on $S_\lambda(U)^{\lambda}_x$, which completes the proof of the theorem. \hfill \square

3.2. The proofs of Theorems A, B, and C. \ We now show how each of Theorems A, B, and C from the introduction follow from 3.1.5.
3.2.1. Proof of Theorem A. We recall the setup. We have a CM field $F$ with maximal totally real subfield $F^+$ and a finite set of finite places $S$ of $F$ containing all places above $p$. We fix a choice of complex conjugation $c \in G_{F^+}$. We are given a continuous absolutely irreducible representation 

$$\rho : \text{Gal}(F(S)/F) \rightarrow \text{GL}_d(E).$$

We assume there is a continuous totally odd character $\mu : G_{F^+} \rightarrow \mathbb{E}^\times$ and an invertible symmetric matrix $P$ such that the pairing $\langle a, b \rangle = aP^{-1}b$ on $E^d$ is perfect and satisfies 

$$\langle \rho(\sigma)a, \rho(\sigma)c \rangle b = \mu(\sigma)(a, b).$$

Since $\rho$ is absolutely irreducible, $P$ is unique up to scalar. The adjoint representation of $\text{Gal}(F(S)/F)$ on $\text{ad}(\rho) = \text{gl}_d(E)$ extends to an action of $\text{Gal}(F(S)/F^+)$ by letting $c$ act by $X \mapsto -P^tXP^{-1}$, and this is independent of the choice of $c$ and of $P$.

By choosing a $\text{Gal}(F(S)/F)$-stable $O$-lattice in $V_\rho$, we may assume that $\rho$ takes values in $GL_d(O)$. The semisimplification of its reduction modulo the maximal ideal of $O$ does not depend on the choice of lattice, and we denote it by $\overline{\rho} : \text{Gal}(F(S)/F) \rightarrow \text{GL}_d(F)$. Assuming there is a finite extension $L/F$ of CM fields, a regular algebraic polarizable cuspidal automorphic representation $\Pi$ of $\text{GL}_d(\mathbb{A}_L)$, and an isomorphism $\iota : \overline{\rho} \rightarrow \Xi$ such that:

(a) $\rho|_{G_L} \otimes \overline{\rho}_p \cong \rho|_{G_F}$;

(b) $\rho_p \notin L$ and $\pi(G_{L(p)})$ is adequate;

we want to show $H_1^i(F(S)/F^+, \text{ad}(\rho)) = 0$.

By 1.3.1, we can define a continuous homomorphism $r : \text{Gal}(F(S)/F^+) \rightarrow G_d(E)$ such that $r|_{G_p} = \rho$ and $\nu \circ r = \mu$, and there is an isomorphism $\text{ad}(r) \cong \text{ad}(\rho)$ of $\text{Gal}(F(S)/F^+)$-representations. We may also assume $r$ takes values in $G_d(O)$ and letting $\tau = r \otimes_O \mathbb{F}$, that $\tau|_{G_p} = \overline{\rho}$. Then 3.1.3 implies that $\tau(G_{L(p)})$ is adequate. Theorem A then follows from 3.1.5. 

3.2.2. Proof of Theorem B. We have a totally real field $F^+$ and a finite set of finite places $S$ of $F^+$ containing all places above $p$. Let $F^+(S)$ be the maximal extension of $F^+$ unramified outside $S$ and all places above $\infty$. Let 

$$\rho : \text{Gal}(F^+(S)/F^+) \rightarrow \text{GL}_d(E)$$

be a continuous, absolutely irreducible representation. We assume that $\rho$ satisfies one of the following:

(GO) $\rho$ factors through a map $\text{Gal}(F^+(S)/F^+) \rightarrow \text{GO}_d(E)$, that we again denote by $\rho$, with totally even multiplier character;

(GSp) $\rho$ factors through a map $\text{Gal}(F^+(S)/F^+) \rightarrow \text{GSp}_d(E)$, that we again denote by $\rho$, with totally odd multiplier character.

We will refer to the former as the GO-case, and the second as the GSp-case. If we are in the GO-case, then we let $\text{ad}(\rho)$ and $\text{ad}^0(\rho)$ denote the Lie algebra $\mathfrak{go}_d(E)$ of $\text{GO}_d(E)$ and sub-Lie algebra $\mathfrak{so}_d(E)$, respectively, with the adjoint action $\text{ad} \circ \rho$ of $\text{Gal}(F^+(S)/F^+)$. If we are in the GSp-case, then we let $\text{ad}(\rho)$ and $\text{ad}^0(\rho)$ denote the Lie algebra $\mathfrak{gsp}_d(E)$ of $\text{GSp}_d(E)$ and sub-Lie algebra $\mathfrak{sp}_d(E)$, respectively, with the adjoint action $\text{ad} \circ \rho$ of $\text{Gal}(F^+(S)/F^+)$.

There is a splitting $\mathfrak{so}_d(E) = \mathfrak{so}_d(E) \oplus E$, resp. $\mathfrak{gsp}_d(E) = \mathfrak{sp}_d(E) \oplus E$, with $E$ the Lie algebra of the centre of $\text{GO}_d(E)$, resp. $\text{GSp}_d(E)$, that is stable under the adjoint action. We thus get a $\text{Gal}(F^+(S)/F^+)$-equivariant splitting $\text{ad}(\rho) = \text{ad}^0(\rho) \oplus E$, where $E$ has the trivial $\text{Gal}(F^+(S)/F^+)$-action. This gives a decomposition 

$$H_2^i(F^+(S)/F^+, \text{ad}(\rho)) = H_2^i(F^+(S)/F^+, \text{ad}^0(\rho)) \oplus H_1^1(F^+(S)/F^+, E).$$

For each $v|p$ in $F^+$, the local group $H_2^i(F^+_v, E) = \ker(H_1^1(F^+_v, E) \rightarrow H_1^1(F^+_v, B_{dR} \otimes \mathbb{Q}_p, E))$ is the one dimensional $E$-subspace of $\text{Hom}(G_v, E)$ corresponding to the unramified extension [BK, Example 3.9]. By class field theory, $H_1^1(F^+_v, E) = 0$. So, we want to show:

1. $H_2^2(F^+(S)/F^+, \text{ad}^0(\rho)) = 0$;
2. $H_2^2(F^+(S)/F^+, \text{ad}^0(\rho)) = 0$;
3. for each $v|p$ in $F^+$, the following natural map is an isomorphism 

$$H^1(F^+(S)/F^+, \text{ad}^0(\rho)) \rightarrow \prod_{v|p} H^1(F^+_v, \text{ad}^0(\rho))/H_1^1(F^+_v, \text{ad}^0(\rho));$$
under the assumption that there is a finite extension $L^+/F^+$ of totally real fields, a regular algebraic polarizable cuspidal automorphic representation $\pi$ of $\GL_d(\A_L^+)$, and an isomorphism $\iota : \Q_p \xrightarrow{\sim} C$ such that:

(a) $\rho|_{L^+} \otimes \Q \cong \rho_{\pi,i}$;
(b) $\overline{\rho}(\GL_{L^+}(\Q_p))$ is adequate.

Choose a CM extension $F/F^+$ such that $F$ is not contained in the subfield of $\mathcal{F}$ fixed by $\overline{\rho}|_{L^+}(\Q_p)$, and set $L = FL^+$. This implies that $\zeta_p \notin L$ and $\overline{\rho}(\GL_L(\Q_p))$ is still adequate, i.e. $\rho|_{G_F}$ satisfies assumption (b) of Theorem A. The closed subgroup $\Gal(F(S)/F^+(S))$ of $\Gal(F(S)/F^+)$ is either trivial or order 2, hence $H^1(F^+/F^+, \ad^0(\rho)) = 0$. Inflation-restriction then gives an isomorphism $H^1(F^+/F^+, \ad^0(\rho)) \cong H^1(F(S)/F^+, \ad^0(\rho))$ for all $i$, and we can replace $F^+/F^+$ with $F(S)/F^+$ in each of 1., 2., and 3. above.

Let $\Pi$ be the base change, using [AC89, Theorem 4.2 of Chapter 3], of $\pi$ to $\GL_d(\A_F)$. The automorphic representation $\II$ is regular algebraic and polarizable, and the fact that $\rho|_{G_L}$ is absolutely irreducible implies that it is cuspidal. Thus, $\rho|_{G_F}$ satisfies assumption (a) of Theorem A. Fix a choice $c$ of complex conjugation. If we are in the GO-case, set $P = \rho(c)$. If we are in the GSp-case, set $P = \rho(c)J_n^{-1}$, where $J_n$ is the matrix defining the symplectic pairing on $E^d$. Then $P$ is an invertible symmetric matrix such that the pairing $\langle a, b \rangle$ on $E^d$ given by $t a P^{-1} b$ satisfies

$$\langle \rho(\sigma)a, \rho(\sigma)c b \rangle = \mu(\sigma) \langle a, b \rangle,$$

for all $\sigma \in G_F$, where $\mu$ is the multiplier character of $\rho$. As in 3.2.1, we can extend the adjoint action of $G_F$ on $\gl_d(E)$ to an action of $G_F^+$ by letting $c$ act as $X \mapsto PXP^{-1}$. With this action, the inclusion $\so_d(E) \to \gl_d(E)$ is $G_F^+$-equivariant if we are in the GO-case, and the inclusion $\sp_d(E) \to \gl_d(E)$ is $G_F^+$-equivariant if we are in the GSp-case. Let $\ad(\rho|_{G_F})$ denote $\gl_d(E)$ with this $G_F^+$-action. Since the $G_F$ action on $\ad(\rho|_{G_F})$ is isomorphic to $\rho|_{G_F} \otimes (\rho|_{G_F})^\vee$ and $\rho|_{G_F}$ is absolutely irreducible, a theorem of Chevellay [Che55, pg. 88] implies $\ad(\rho|_{G_F})$ is a semisimple $\Gal(F(S)/F^+)$-representation, which implies that it is semisimple as a $\Gal(F(S)/F^+)$-representation. Thus, $\ad^0(\rho)$ is a $\Gal(F(S)/F^+)$-equivariant direct summand of $\ad(\rho|_{G_F})$. The result now follows from Theorem A.

3.2.3. Proof of Theorem C. We recall the setup. Fix a continuous homomorphism

$$\tau : \Gal(F(S)/F^+) \to \G_d(F)$$

inducing an isomorphism $\Gal(F/F^+) \xrightarrow{\sim} \G_d(F)/\G_d(F^+)$, and a continuous, totally odd, deRham character $\mu : \Gal(F(S)/F^+) \to \mathcal{O}^\times$ with $\nu \circ \tau = \mu \mod \mathfrak{m}_\nu$. We assume $\tau|_{G_F}$ is absolutely irreducible. The ring $R_S$ in the statement of Theorem C is the universal type $S$ deformation ring for the global deformation datum

$$S = (F/F^+, S, \emptyset, \mathcal{O}, \tau, \mu, \emptyset).$$

Theorem C then follows immediately from 3.1.5 and 1.3.13.

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