DEFORMATIONS OF HILBERT MODULAR GALOIS REPRESENTATIONS AND 
ADJOINT SELMER GROUPS

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Abstract. We prove the vanishing of the geometric Bloch–Kato Selmer group for the adjoint representation 
of a Galois representation associated to a Hilbert modular form under mild assumptions on the residual 
image. In particular, we do not assume the residual image satisfies the Taylor–Wiles hypothesis. Using 
this, we deduce that the localization and completion of the universal deformation ring for the residual 
representation at the characteristic zero point induced from the Hilbert modular form is formally smooth of 
the correct dimension. We do this by employing the Taylor–Wiles patching strategy in a way inspired by the 
work of Skinner–Wiles to get around the Taylor–Wiles hypothesis. Along the way we give a characterization 
of smooth closed points on the generic fibre of Kisin’s potentially semistable deformation in terms of their 
Weil–Deligne representations.

Introduction

Consider a number field $F$ and a finite set of places $S$ of $F$ containing all those above a fixed rational 
prime $p$ and $\infty$, and let $G_{F,S}$ denote the Galois group of the maximal extension of $F$ unramified outside of 
$S$. Given a $p$-adic representation $V$ of $G_{F,S}$, Bloch and Kato [BK] defined certain subspaces $H^1_{f}(G_{F,S},V) \subseteq H^1_{g}(G_{F,S},V) \subseteq H^1(G,V)$ of the Galois cohomology group $H^1(G,V)$, knowns as the 
Bloch–Kato Selmer group and geometric Bloch–Kato Selmer group, respectively. If $V$ is deRham, resp. crystalline, then $H^1_{g}(G_{F,S},V)$, resp. $H^1_{f}(G_{F,S},V)$, is the 
subspace of $H^1(G,V) = \text{Ext}^1_{\mathbb{Q}_p[G_{F,S}]}(\mathbb{Q}_p,V)$ of extensions of the trivial representation by $V$ that are deRham, resp. crystalline. They then made a far reaching and influential conjecture that relates the dimension of 
$H^1_{f}(G_{F,S},V)$ to the order of vanishing of the $L$-function of the dual representation of $V$ at the point $s = 1$. 
One prediction of this conjecture is that if the representation $V$ is pure of motivic weight zero, then 
$H^1_{f}(G_{F,S},V) = H^1_{g}(G_{F,S},V) = 0$. 
This is in accordance with a philosophy of Grothendieck that in a conjectural category of mixed motives, 
there should be no nontrivial extensions of pure motives of the same weight.

Let $E$ be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. Given any continuous pure representation 
$\rho : G_{F,S} \rightarrow \text{Aut}_E(V)$ unramified outside finitely many primes on a finite dimensional $E$-vector space $V$, one naturally obtains a 
pure weight zero representation called the adjoint representation 
$\text{ad}(\rho) = \text{Hom}_E(V,V)$ by letting $\sigma \in G_{F,S}$ act on an endomorphism $f$ by $\sigma f = \rho(\sigma) \circ f \circ \rho(\sigma)^{-1}$, and the Bloch–Kato conjecture predicts 
$H^1_{g}(G_{F,S},\text{ad}(\rho)) = 0$.

In the case where $\rho$ is the representation arising from an elliptic curve over $\mathbb{Q}$, this prediction was first proved 
by Flach [Fla92] by a method using Euler systems, assuming that the elliptic curve has good reduction at $p$, 
that $p \geq 5$, and that the associated residual representation 
$\bar{\rho} : G_{Q,S} \rightarrow \text{GL}_2(\mathbb{F}_p)$ 
representation is surjective. A corollary of the breakthrough work of Wiles and Taylor–Wiles is this vanishing 
in the case that $\rho$ is the representation coming from a modular form of weight $k \geq 2$ and level $\Gamma_1(N)$ with 
$p > 2$, $p^2 \nmid N$, and such that the residual representation satisfies the Taylor–Wiles hypothesis: that the 
restriction of $\bar{\rho}$ to $G_{F(\zeta_p)}$ is absolutely irreducible, where $\zeta_p$ is a primitive $p$th rood of unity. This results
from their so-called $R = \mathbb{T}$ theorem that equates a certain universal deformation ring of $\rho$ to a Hecke algebra. This is because the tangent space of the deformation ring they consider at the characteristic zero point corresponding to the modular form is equal to the Bloch–Kato Selmer group, while the tangent space of the Hecke algebra at that point is trivial, since the Hecke algebra is reduced. After some intermittent work, Kisin [Kis04] showed the vanishing of $H^1_g(G_{\mathbb{Q},s}, \text{ad}(\rho))$ for modular forms of weight $k \geq 2$ and arbitrary level, assuming only a mild condition on the residual representation. The method uses some of the ideas of Taylor–Wiles coupled with a careful analysis of the integral étale cohomology of modular curves. We mention also the result of Weston [Wes04] which applies to non-CM forms with certain hypotheses on the level, but has no restriction on the residual representation.

In the case that $\rho$ is the Galois representation associated to a Hilbert modular form, one can deduce results of this form from the $R[1/p] = \mathbb{T}[1/p]$ theorems [Kis09a, Theorem 3.4.11] and [KW09, Propositions 9.2 and 9.3] whenever the assumptions of those theorems are satisfied. It is not hard to see that if $\rho$ is absolutely irreducible, to prove $H^1_g(G_{F,S}, \text{ad}(\rho)) = 0$, one may replace $F$ with a finite extension $L/F$ such that $\rho|_{G_L}$ is absolutely irreducible. Using this observation, [Kis09a, Theorem 3.4.11] implies that $H^1_g(G_{F,S}, \text{ad}(\rho)) = 0$ assuming $p > 2$, the Hilbert modular form has parallel weight 2, and that the residual representation satisfies the Taylor–Wiles hypothesis as well an additional assumption if $p = 5$ (see [Kis09a, Assumption (3) of Theorem 3.5.5]).

Many of the subsequent modularity lifting theorems are of the form $R^{\text{red}} = \mathbb{T}$, and so do not imply vanishing of $H^1_g(G_{F,S}, \text{ad}(\rho))$. We observe that one can still use the Taylor–Wiles patching method to deduce $H^1_g(G_{F,S}, \text{ad}(\rho)) = 0$ for modular Galois representations in all regular weights, provided one knows that modular points on local deformation rings are smooth. Indeed, the method yields a ring $R_\infty$ and a module $M_\infty$, and a control theorem that relates them to our deformation ring and a space of cusp forms. The most subtle point in proving modularity lifting theorems is to understand the components of $R_\infty$ and how they relate to congruences between modular forms. But if we are only interested in the infinitesimal deformation theory of the characteristic zero point coming from $\rho$, we can localize and complete at this point, and if we know that $\rho$ determines a smooth point on the local deformation rings, it also determines a smooth point on $R_\infty$. Then we can apply the Auslander–Buchsbaum formula to the completion and deduce that the localized and completed deformation ring acts freely on a finite dimensional vector space of cusp forms, from which we can deduce $H^1_g(G_{F,S}, \text{ad}(\rho)) = 0$. In this strategy, because we are only interested in the localization and completion of our deformation ring at the prime corresponding to the fixed characteristic zero modular Galois representation, we can take this a step further and carry out the patching argument in a way inspired by the work of Skinner–Wiles [SW00, SW01] to remove the Taylor–Wiles hypothesis in most cases. The main theorem is the following.

**Theorem A.** Let $F$ be a totally real field and $\mathbb{A}_F$ the adeles of $F$. Let $\pi$ be a regular algebraic cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$. Fix a rational prime $p$, and an isomorphism $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_p$. Let $S$ denote a finite set of places of $F$ containing all places above $p$ and all places above $\infty$. Let

$$\rho_{\pi,\iota} : G_{F,S} \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$$

be the Galois representation associate to $\pi$ and $\iota$, let $\text{ad}(\rho_{\pi,\iota})$ denote its adjoint representation, and let $\overline{\rho}_{\pi,\iota}$ denote its semisimple residual representation.

Assume the following:

(a) $\overline{\rho}_{\pi,\iota}$ is absolutely irreducible;

(b) if $\pi$ has complex multiplication by a CM extension $L/F$, then $p > 2$ and $L \nsubseteq F(\zeta_p)$.

Then the geometric Bloch–Kato Selmer group

$$H^1_g(G_{F,S}, \text{ad}(\rho_{\pi,\iota})) := \ker \left( H^1(G_{F,S}, \text{ad}(\rho_{\pi,\iota})) \rightarrow \prod_{v | p} H^1(G_v, B_{\text{dR}} \otimes_{\mathbb{Q}_p} \text{ad}(\rho_{\pi,\iota})) \right)$$

is trivial.

In the statement above $\zeta_p$ denote a primitive $p$th root of unity, $G_v$ denotes a choice of decomposition group at $v | p$, and $B_{\text{dR}}$ is Fontaine’s ring of deRham periods. As noted above, we are permitted to replace $F$ with (suitable) finite extensions, so the same conclusion holds assuming potential modularity of the Galois representation (see 3.3.1). The assumption (b) is a characteristic zero Taylor–Wiles condition. Although the patching method used is inspired by [SW00], the author is currently unable to handle residually reducible
representations due to the subtlety in the relation between the universal deformation ring and the universal pseudodeformation ring (see Remark 3.2.10).

As mentioned above, an important step is to show that the points on the local deformation rings coming from regular algebraic cuspidal automorphic representations are smooth. By local-global compatibility, the Weil–Deligne representations attached to the local factors of a regular algebraic cuspidal representations satisfy the property that they do not admit any nontrivial morphisms to their Tate twist. We say below (1.2.2) that such Weil–Deligne representations are generic. A result of Gee [Gee11, Theorem 2.1.6] coupled with a straightforward calculation using Galois cohomology and obstruction theory shows that if $K$ is a finite extension of $\mathbb{Q}_\ell$ with $\ell \neq p$, then the smooth closed points the generic fibre of the framed deformation ring of a fixed mod $p$ residual representation are precisely the ones whose Weil–Deligne representation satisfy this genericity hypothesis (as observed for instance in [BLGGT14, Lemma 1.3.2]). It was noted in [Cal12, Lemma 2.6] that a related but stronger condition was sufficient to guarantee a point on the generic fibre of Kisin’s potentially semistable deformation ring is smooth (even after any finite base change). We show below that a closed point on the generic fibre of Kisin’s potentially semistable deformation ring is smooth if and only if the associated Weil–Deligne representation is generic (1.3.7 below), the same condition as in the $\ell \neq p$ case. The author finds this a pleasing instance of a sort of “independence of $p$” phenomena for deformation rings. In fact, for the proof of Theorem A, it would suffice to use the results proved in [Kis09b, (A.1)], and it also suffices to know only the direction “generic implies smooth” (as in [BLGGT14, Lemma 2.6] and [Cal12, Lemma 2.6]), but the author has decided to include the complete description of smooth closed points in arbitrary dimension here because he finds it interesting, and to allow for ease of reference later.

As in [Kis04] we use Theorem A to deduce smoothness of the universal deformation ring at automorphic points.

**Theorem B.** Let $p$ be a prime and let $O$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. Let \[ \rho : G_{F,S} \to \text{GL}_2(\mathbb{F}) \]
be a continuous, absolutely irreducible representation. Let $R_{\rho}$ be the universal deformation ring for $\rho$ in the category of complete Noetherian local $O$-algebras with residue field $\mathbb{F}$. Let $x$ be a closed point of $\text{Spec}R_{\rho}[1/p]$ with residue field $E$, and let \[ \rho_x : G_{F,S} \to \text{GL}_2(E) \]
be the pushforward of the universal deformation of $\rho$ via $R_{\rho}[1/p] \to E$.

Assume there is a regular algebraic cuspidal automorphic representation $\pi$ of $\text{GL}_2(\mathbb{A}_F)$ and an isomorphism $\iota : \mathbb{C} \cong \mathbb{Q}_p$ such that $\rho_x \cong \rho_{x,\iota}$. If $\pi$ has complex multiplication by a CM extension $L/F$, we assume $p > 2$ and $L \not\subseteq F(\zeta_p)$.

Then the localization and completion $(R_{\rho})_x^\wedge$ is formally smooth over $E$ of dimension $1 + \delta_F + 2[F : \mathbb{Q}]$, where $\delta_F$ denotes the Leopoldt defect for $F$ and $p$.

As with Theorem A, the conclusion of Theorem B also holds assuming only potential automorphy (see 3.4.1). Part of the interest in proving Theorems A and B in cases with degenerate residual image come from applications of Theorem B to “big $R$ equals big $T$” theorems. In Böckle’s strategy for proving these theorems [Böck01], one uses a “small $R$ equals small $T$” theorem to show that every irreducible component of the universal deformation ring contains a smooth modular point, and then applies the infinite fern of Gouvea–Mazur to show the modular points are dense in each component. A possible application of Theorem B is to use it in conjunction with the modularity lifting results of [SW01] and [Tho14b] to establish “big $R$ equals big $T$” theorems in cases when the Taylor–Wiles hypothesis fails, and then to apply Emeret’s local-global compatibility in the $p$-adic Langlands program to deduce new cases of the Fontaine–Mazur conjecture. The author hopes to report on this in future work.

One can also use Theorem A to deduce formal smoothness of the localization and completion of other “large” deformation rings when the conclusion of Theorem A applies (see 3.4.3). One such example is the big ordinary (or nearly ordinary) deformation ring. Hida has produced produced interesting formulae for adjoint $L$-invariants in $p$-adic families of modular forms conditional on such formal smoothness results (see [Hid04, Theorem 1.1] for the elliptic modular case and [Hid09, Theorem 1.2] for the case of totally real fields). Thus, the results above help establish more cases of these formulae for totally real fields.

It will be apparent to readers familiar with automorphy lifting theorems that these ideas can also be applied in higher dimensions. Namely, to prove vanishing of $H^1_g(G_{F^+},S,\text{ad}(\rho))$, where $F^+$ is a totally
real field, $F/F^+$ is a CM extension, and $\rho$ is the Galois representation associated to a regular algebraic, essentially conjugate self-dual, cuspidal automorphic representation of $\text{GL}_n(k_F)$; at least when the residual representation is adequate when restricted to the cyclotomic subfield. However, using the ideas of [Tho14a], it may be possible to prove such results with weaker assumptions on the residual image. The author hopes to report on this in the future.

**Outline.** We now discuss the organization of this paper. In §1, we recall and prove the relevant facts regarding the deformation theory of Galois representations. The material in §1.1 is standard, except for a small result that may be of independent interest (1.1.11 and 1.1.13) that relates fixed determinant deformation rings with nonfixed determinant deformation rings valid even when the residual characteristic divides the dimension, when the usual trick of taking a root of the universal character does not apply. We then recall in §1.2 the construction of Weil–Deligne representations attached to local Galois representations, and their connection with smooth admissible representations of $\text{GL}_d$. In §1.3 we treat the local theory of Galois deformations with an emphasis on describing the smooth points in the generic fibre of local deformation rings. The global theory is discussed in §1.4, where we recall the connection with the Bloch–Kato Selmer group, and prove some lemmas about the minimal number of generators of certain primes ideals that are necessary for the patching argument. We discuss the automorphic theory necessary for the patching argument in §2; everything here is standard. In §3, we first prove the abstract patching argument in §3.1, and in §3.2 we show the existence of the relevant Taylor–Wiles primes, closely following [Kis04, §6]. In §3.3 and §3.4, we prove Theorems A and B, respectively.

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**Notation**

If $F$ is a number field and $v$ is a place of $F$, we denote by $F_v$ the completion of $F$ at $v$, and if $v$ is non-Archimedean we denote the ring of integers by $\mathcal{O}_{F_v}$.

If $K$ is any field with a fixed algebraic closure $\overline{K}$, we denote by $G_K$ the absolute Galois group $\text{Gal}(\overline{K}/K)$. If $K$ is a local field, we denote by $I_K$ the inertia subgroup, by $W_K$ the Weil group, and by $\text{Frob}_K$ the geometric Frobenius in $G_K/I_K \cong W_K/I_K$. In the case that $K$ is the completion of a number field $F$, we write $G_F$, $I_F$, and $\text{Frob}_F$, for $G_{F_v}$, $I_{F_v}$, and $\text{Frob}_{F_v}$, respectively. If $F$ is a number field and $S$ is a finite set of places of $F$, we let $G_{F,S}$ denote the quotient of $G_F$ corresponding to the maximal subextension of $\overline{F}/F$ unramified outside of places in $S$. A CM extension of a totally real field is always assumed to be imaginary.

If $K$ is a non-Archimedean local field, we let $\text{Art}_K : K^\times \xrightarrow{\sim} W_K$ be the Artin reciprocity map normalized so that uniformizers are sent to geometric Frobenius elements. We normalize the isomorphism of global class field theory compatibly. We denote by $\epsilon$ the $p$-adic cyclotomic character. We use covariant $p$-adic Hodge theory, and normalize our Hodge–Tate weights so that the Hodge–Tate weight of $\epsilon$ is $-1$. For any $d \geq 1$, we let $\text{rec}_K$ be the Local Langlands reciprocity map that takes an irreducible admissible representation of $\text{GL}_d(K)$ to a Frobenius semi-simple Weil–Deligne representation, normalized as in [HT01] and [Hen00]. We then let $\text{rec}_K^T$ be given by $\text{rec}_K^T(\pi) = \text{rec}_K(\pi \otimes | \cdot |^{\frac{1}{d}})$.

If $\iota : K \xrightarrow{\sim} L$ is an isomorphism of fields, and $r : G \to \text{Aut}_K(V)$ is a representation of a group $G$ on a $K$-vector space $V$, then we will denote by $\iota_* r$ the representation of $G$ on the $L$-vector space $V \otimes_{K, \iota} L$.

If $G$ is a group, $A$ is a commutative ring, and $\rho : G \to \text{GL}_n(A)$ is a homomorphism, then we will let $V_\rho$ denote the representation space of $\rho$, i.e. $V_\rho = A^n$ with the $A[G]$-module structure coming from $\rho$.

If $A$ is a commutative local ring, we will denote by $\mathfrak{m}_A$ its maximal ideal. If $A$ is a commutative ring and $x : A \to D$ is a homomorphism with $D$ a domain, then we denote by $A_x$ the localization of $A$ at $\ker(x)$, and $A_x^\wedge$ the localization and completion of $A$ at $\ker(x)$. If $A$ is a commutative ring and $x \in \text{Spec}A$ has residue field $k_x$, we again denote by $x$ the map $x : A \to k_x$. 


Recall that if $k$ is a field, $R$ is a commutative $k$-algebra, and $x \in \text{Spec}R$, we say $R$ is formally smooth over $k$ at $x$ if there is an open subset $U \subseteq \text{Spec}R$ such that $k \to R_y$ is formally smooth for all $y \in U$. If $R$ is Noetherian and $k$ has characteristic 0, this is equivalent to $R_y$ being regular for all $y \in U$. If $R$ is further excellent, the regular locus is open, so $R$ is formally smooth at $x$ if and only if $R_x$ is regular, which happens if and only if $k \to R_x$ is formally smooth, which happens if and only if $R_x^\wedge$ is isomorphic to a power series ring over its residue field. We will frequently use these equivalences without comment.

If $\Lambda$ is a complete Noetherian local commutative ring with residue field $k$, we let $\text{CNL}_{\Lambda}$ be the category whose objects are complete Noetherian local commutative $\Lambda$-algebras $A$ such that the structure map $\Lambda \to A$ induces an isomorphism $k \to A/\mathfrak{m}_A$, and whose morphisms are local $\Lambda$-algebra morphisms. We will refer to an object in $\text{CNL}_{\Lambda}$ as a $\text{CNL}_{\Lambda}$-algebra and a morphism in $\text{CNL}_{\Lambda}$ as a $\text{CNL}_{\Lambda}$-morphism.

If $G$ is a topological group, and $M$ is a topological $G$-module, the cohomology groups $H^i(G, M)$ are always assumed to be the continuous cohomology groups, i.e. the cohomology groups computed with continuous cochains.

1. Deformation theory

We first set up some notation and conventions that will be used throughout this section.

We let $E$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. We fix a uniformizer $\varpi$ of $\mathcal{O}$.

Given a group $G$ and a character $\psi : G \to \Lambda$, we will often again denote by $\psi$ the $A$-valued character, for any $\text{CNL}_{\Lambda}$-algebra $A$, given by composing $\psi$ with the homomorphism $\Lambda^\times \to A^\times$ coming from the $\Lambda$-algebra structure on $A$.

Given a $\text{CNL}_{\Lambda}$-algebra $A$, we will denote by $\text{Spf}A$ the element of the opposite category $\text{CNL}_{\Lambda}^{\text{op}}$ represented by $A$. Since the formal scheme associated to $A$ is uniquely determined by the set valued functor $\text{Hom}_{\text{CNL}_{\Lambda}}(A, \cdot)$ on $\text{CNL}_{\Lambda}$, this should cause no confusion. Below, fibre products of elements on $\text{CNL}_{\Lambda}$ are always taken in the category $\text{CNL}_{\Lambda}$, so $\text{Spf}A \times \text{Spf}B$ will denote $\text{Spf}(A \otimes_{\Lambda} B)$. If $X$ is an element of $\text{CNL}_{\Lambda}^{\text{op}}$ and $A$ is a $\text{CNL}_{\Lambda}$-algebra, we will denote by $X_A$ the fibre product $X \times \text{Spf}A$.

If $D$ is a finite module over a commutative ring $A$, and $\{\text{Fil}_D^j\}_{i \in \mathbb{Z}}$ is a decreasing separated exhaustive filtration on $D$ by $A$-submodules, we sometimes write $D^+$ for $\text{Fil}_D^0$. We let $\text{ad}(D) = \text{End}_A(D)$ with the decreasing, separated, exhaustive filtration $\{\text{Fil}_{\text{ad}(D)}^j\}_{i \in \mathbb{Z}}$ given by $\text{Fil}_{\text{ad}(D)}^j = \{f \in \text{End}_A(D) \mid f(\text{Fil}_D^j) \subseteq \text{Fil}_D^j\}$ for all $j \in \mathbb{Z}$.

1.1. Generalities. Let $G$ be a pro-finite group satisfying the $p$-finiteness condition: for any open subgroup $H$ of $G$, the set of continuous homomorphisms from $H$ to $\mathbb{F}_p$ is finite. Let $\Lambda$ be a complete Noetherian local commutative ring with residue field $k$. Throughout this section, $k$ is assumed to be either a finite extension of $\mathbb{F}_p$ or a finite extension of $\mathbb{Q}_p$. Let $\rho : G \to \text{GL}_d(A)$ be a representation of $G$ with $A$ a $\text{CNL}_{\Lambda}$-algebra. If $k$ is finite, we say $\rho$ is continuous if is is continuous with respect to the $\mathfrak{m}_A$-adic topology on $A$. If $k$ is a finite extension of $\mathbb{Q}_p$, then $A$ is canonically a $k$-algebra, and we say $\rho$ is continuous if $\rho \bmod \mathfrak{m}_A^n : G \to \text{GL}_d(A/\mathfrak{m}_A^n)$ is continuous for all $n \geq 1$, where we give $A/\mathfrak{m}_A^n$ the topology as a finite dimensional $k$-vector space.

Fix a continuous representation $\overline{\rho} : G \to \text{GL}_d(k)$.

Definition 1.1.1. For $A$ in $\text{CNL}_{\Lambda}$, a lift of $\overline{\rho}$ to $A$ is a continuous representation $\rho : G \to \text{GL}_d(A)$ such that $\overline{\rho}_A \bmod \mathfrak{m}_A$ is equal to $\overline{\rho}$. We say that two lifts $\rho$ and $\rho'$ to $A$ are equivalent if there is $g \in \ker(\text{GL}_d(A) \to \text{GL}_d(k))$ with $g \rho g^{-1} = \rho'$. A deformation of $\overline{\rho}$ to $A$ is an equivalence class of lifts.

For a finite set $T$, a $T$-framed lift to $A$ is a tuple $(\rho, (\alpha_v)_{v \in T})$ where $\rho$ is a lift of $\overline{\rho}$ to $A$ and $\alpha_v \in \ker(\text{GL}_d(A) \to \text{GL}_d(k))$. We say two $T$-framed lifts $(\rho, (\alpha_v)_{v \in T})$ and $(\rho', (\alpha_v')_{v \in T})$ to $A$ are equivalent if there is $g \in \ker(\text{GL}_d(A) \to \text{GL}_d(k))$ such that $g \rho g^{-1} = \rho'$ and $g \alpha_v = \alpha'_v$ for each $v \in T$. A $T$-framed deformation of $\overline{\rho}$ to $A$ is an equivalence class of lifts.

If $\rho$ is a lift of $\overline{\rho}$, resp. $(\rho, (\alpha_v))$ is a $T$-framed lift of $\overline{\rho}$, we will denote by $[\rho]$, resp. $[\rho, (\alpha_v)]$ the corresponding deformation, resp. $T$-framed deformation. However, we will often abuse notation by denoting a deformation, or a $T$-framed deformation, by an element in its equivalence class. Note that if $T = \emptyset$, then a $T$-framed lift, resp. a $T$-framed deformation, is simply a lift, resp. a deformation.
Definition 1.1.2. The lifting functor for \( \mathfrak{p} \) is the set valued functor on CNL\(_A\)
\[
D^\square_{\mathfrak{p}}(A) = \text{the set of lifts of } \mathfrak{p} \text{ to } A.
\]
If \( D^\square_{\mathfrak{p}} \) is representable, we call the representing object the universal lifting ring for \( \mathfrak{p} \), and denote it by \( R^\square_{\mathfrak{p}} \).

The deformation functor for \( \mathfrak{p} \) is the set valued functor on CNL\(_A\)
\[
D_{\mathfrak{p}}(A) = \text{the set of deformations of } \mathfrak{p} \text{ to } A.
\]
If \( D_{\mathfrak{p}} \) is representable, we call the representing object the universal deformation ring for \( \mathfrak{p} \), and denote it by \( R_{\mathfrak{p}} \).

Let \( T \) be a finite set. The \( T \)-framed deformation functor for \( \mathfrak{p} \) is the set valued functor on CNL\(_A\)
\[
D^{\square_T}_{\mathfrak{p}}(A) = \text{the set of } T \text{-framed lifts of } \mathfrak{p} \text{ to } A.
\]
If \( D^{\square_T}_{\mathfrak{p}} \) is representable, we call the representing object the universal \( T \)-framed deformation ring for \( \mathfrak{p} \), and denote it by \( R^{\square_T}_{\mathfrak{p}} \).

Remark 1.1.3. In fact the functors \( D^\square_{\mathfrak{p}} \) and \( D_{\mathfrak{p}} \) are special cases of \( D^{\square_T}_{\mathfrak{p}} \). Indeed, if \( T = \emptyset \), then \( D^{\square_T}_{\mathfrak{p}} = D_{\mathfrak{p}} \).

If \( T \) is a singleton, then the morphism of functors \( D^\square_{\mathfrak{p}} \to D^{\square_T}_{\mathfrak{p}} \) given by \( \rho \mapsto [\rho, 1] \) is an isomorphism with inverse \( [\rho, \alpha] \mapsto \alpha^{-1} \rho \alpha \) (which is independent of the choice of representative \( (\rho, \alpha) \) in \([\rho, \alpha]\)).

In some references a deformation of \( V_{\mathfrak{p}} \) to a CNL\(_A\)-algebra \( A \) is defined as being a pair \((V_A, \phi_A)\), where \( V_A \) is a free \( A \)-module of rank \( n \) with a continuous \( A \)-linear \( G \)-action, and \( \phi : V_A \otimes_A k \xrightarrow{\sim} V_{\mathfrak{p}} \) is a \( k[G] \)-isomorphism. In these references, a \( T \)-framed deformation is defined as being a tuple \((V_A, \phi_A, (\beta_v)_{v \in T})\), where \((V_A, \phi_A)\) is a deformation and \( \beta_v \) is a choice of basis for \( V_A \) for each \( v \in T \) such that \( \phi_A \) takes \( \beta_v \) \( \mod m_A \) to our fixed basis for \( V_{\mathfrak{p}} \). With these definitions, set-valued functors \( D_{\mathfrak{p}} \) and \( D^{\square_T}_{\mathfrak{p}} \) on CNL\(_A\) are then defined by letting \( D_{V_{\mathfrak{p}}}(A) = \text{the set of isomorphism classes of deformations of } V_{\mathfrak{p}} \text{ to } A \), and \( D^{\square_T}_{V_{\mathfrak{p}}}(A) = \text{the set of isomorphism classes of } T \text{-framed deformations of } V_{\mathfrak{p}} \text{ to } A \). It is easy to see that there are isomorphisms of functors \( D_{V_{\mathfrak{p}}} \cong D_{\mathfrak{p}} \) and \( D^{\square_T}_{V_{\mathfrak{p}}} \cong D^{\square_T}_{\mathfrak{p}} \), so results in the literature proved for \( D_{V_{\mathfrak{p}}} \) and \( D^{\square_T}_{V_{\mathfrak{p}}} \) apply to \( D_{\mathfrak{p}} \) and \( D^{\square_T}_{\mathfrak{p}} \) (and vice versa).

The following foundational result is well known. It can be proved using Schlessinger’s criterion as in [Maza, §1.2], or more directly using [Gou, Appendix 1] as in the proof of [CHT08, Proposition 2.2.9].

Theorem 1.1.4. If \( T \neq \emptyset \) or \( \text{End}_{k[G]}(V_{\mathfrak{p}}) = k \), then \( D^{\square_T}_{\mathfrak{p}} \) is representable. In particular (see 1.1.3), \( D^\square_{\mathfrak{p}} \) is representable, and \( D_{\mathfrak{p}} \) is representable if \( \text{End}_{k[G]}(V_{\mathfrak{p}}) = k \).

If \( A \) is the ring of integers of a finite extension of \( \mathbb{Q}_p \) (hence \( k \) is finite), then for any CNL\(_A\)-algebra \( A \), any closed point of \( A[1/p] \) has residue field a finite extension of \( A[1/p] \) (see [KW09, Proposition 2.2(i)], for example). It follows that if \( \rho : G \to \text{GL}_d(A) \) is a lift of \( \mathfrak{p} \) and \( x \) is a closed point of \( A[1/p] \) with residue field \( E \), then the pushforward \( \rho_x : G \to \text{GL}_d(E) \) of \( \rho \) via \( x : A[1/p] \to E \) is a continuous representation. The same holds for the pushforward \( \rho_x \) of \( \rho \) via any \( A[1/p] \)-algebra homomorphism \( x : A[1/p] \to E \) with \( E \) an algebraic extension of \( \mathbb{Q}_p \). We will often use this fact bellow without comment.

The proof of our main theorems will rely crucially on Kisin’s method for analysing the generic fibre of universal deformation rings, the linchpin of which is the following result of Kisin.

Theorem 1.1.5. Assume \( \Lambda \) is the ring of integers of a finite extension of \( \mathbb{Q}_p \) and \( E \) is a totally ramified extension of \( \Lambda[1/p] \) (hence \( k = \mathbb{F} \)). Let \( T \) be a finite set, and assume either \( T \neq \emptyset \) or \( \text{End}_{k[G]}(V_{\mathfrak{p}}) = \mathbb{F} \). Let \( \rho : G \to \text{GL}_d(O) \) be a lift of \( \mathfrak{p} \), and let \( \rho_E = \rho \otimes_{\mathbb{Q}_p} E \). Let \( x : R^{\square_T}_{\mathfrak{p}} \to O \) denote the CNL\(_A\)-morphism induced by \( \rho(1) \in T \).

Then \( T \)-framed deformation functor \( D^{\square_T}_{R_E} \) on CNL\(_E\) is represented by the localization and completion of \( R^{\square_T}_{\mathfrak{p}} \otimes_{\Lambda} E \) at the point \( x \otimes 1 \).

Proof. This is [Kis09a, Lemma 2.3.3 and Proposition 2.3.5]. It is assumed there that \( G \) is the absolute Galois group of a local field and \( T \) is either empty for a singleton, but the proof carries over in our level of generality. In fact, [Kis09a, Proposition 2.3.5] goes further by identifying certain groupoids, which together with 1.1.4 implies what we want (see [Kis09a, §A.5]).

Definition 1.1.6. Assume \( D^{\square_T}_{\mathfrak{p}} \) is representable. We call a subfunctor \( D \subseteq D^{\square_T}_{\mathfrak{p}} \) a deformation problem if
1. \( \mathcal{D} \) is representable (by a quotient of \( R^{\square}_{\mathfrak{p}} \));

2. for any CNL\( \Lambda \)-algebra \( A \), if two elements in \( D^{\square,r}_{\mathfrak{p}}(A) \) have the same image in \( D_{\mathfrak{p}}(A) \), then one is in \( \mathcal{D}(A) \) if and only if the other is.

We will call a quotient of \( R^{\square}_{\mathfrak{p}} \) that represents a deformation problem a deformation quotient.

The reader may wish to compare this definition with [Mazb, §23] and [CHT08, Definition 2.2.2] (see also [CHT08, Lemma 2.2.3]). One could modify this definition to include the case that \( D^{\square,r}_{\mathfrak{p}} \) is not representable by replacing 1. with the notion of relative representability (see [Mazb, §19]), but we will not need this.

**Proposition 1.1.7.** Assume \( T \neq \emptyset \). Let \( C \) denote the centralizer of the image of \( \mathfrak{p} \) in \( M_{d \times d}(k) \). Let \( D^{\square,r} \subseteq D^{\square,r}_{\mathfrak{p}} \) be a deformation problem and let \( \mathcal{D} \) denote its image in \( D_{\mathfrak{p}} \). The map of functors \( \mathcal{D}^{\square,r} \to \mathcal{D} \) is formally smooth of relative dimension \( d^{|T|} - \dim_k C \).

**Proof.** It is straightforward to check the lifting criterion for formal smoothness, because we can always lift elements of \( \ker(\text{GL}_d(A/I) \to \text{GL}_d(k)) \) to \( \ker(\text{GL}_d(A) \to \text{GL}_d(k)) \). The relative dimension can be checked on tangent spaces, and it is also straightforward to check that the relative tangent space \( \ker(D^{\square,r}(k[\varepsilon]) \to \mathcal{D}(k[\varepsilon])) \) is isomorphic to \( M_{d \times d}(k[T])/C \).

1.1.8. One standard deformation condition of which we will make use is that of a fixed determinant. Let \( \psi : G \to \Lambda^* \) be a continuous character with \( \psi \mod \mathfrak{m}_\Lambda = \det(\mathfrak{p}) \). We let \( D^{\square,r,\psi}_{\mathfrak{p}} \subseteq D^{\square,r}_{\mathfrak{p}} \) be the subfunctor of \( T \)-framed deformations \( [\rho, (\alpha_v)] \) such that \( \det(\rho) = \psi \). This clearly satisfies 2. of 1.1.6, and it is easy to see that it satisfies 1. as we simply need to mod out by the ideal generated by \( \{\det(\rho^\Lambda(\sigma)) - \psi(\sigma) \mid \sigma \in G\} \) in \( R^{\square}_{\mathfrak{p},r} \), where \( \rho^\Lambda \) denotes any choice of lift in the universal \( T \)-framed deformation. From this description and 1.1.5, it also follows that (in the notation of 1.1.5) if \( \det(\rho) = \psi \), then \( D^{\square,r,\psi}_{\mathfrak{p}} \) is represented by the localization and completion of \( R^{\square,r,\psi}_{\mathfrak{p}} \otimes \Lambda E \) at the point \( x \otimes 1 \).

For any finite dimensional \( k \)-vector space \( M \) with a continuous \( G \)-action, we let \( Z^i(G, M) \) be the \( k \)-vector space of continuous \( i \)-cocycles and \( H^i(G, M) \) the \( i \)-th (continuous) cohomology group. Since \( G \) satisfies the \( p \)-finiteness condition, \( H^1(G, M) \) and \( Z^1(G, M) \) are finite dimensional. If either \( G = G_K \) for \( K \) a non-Archimedean local field, or \( G = G_{F,S} \) for \( F \) a number field and \( S \) a finite set of places of \( F \), then \( H^2(G, M) \) is also finite dimensional. Let \( \text{ad}(\mathfrak{p}) = M_{d \times d}(k) \) with the adjoint \( G \)-action. Let \( \text{ad}^0(\mathfrak{p}) \) denote its trace zero subspace, and let \( \mathfrak{z} \) denote the subspace of scalar elements. If \( d \) is invertible in \( k \), then \( \text{ad}(\mathfrak{p}) = \text{ad}^0(\mathfrak{p}) \oplus \mathfrak{z} \) as \( k[G] \)-modules. The next proposition relates the relative tangent spaces and dimensions of the universal lifting and deformation rings to group cohomology. Before stating it we note that even when \( D_{\mathfrak{p}} \) is not representable, the set \( D(k[\varepsilon]) \) still has the structure of a \( k \)-vector space (see [Mazb, §16]).

**Proposition 1.1.9.** Write \( H^1(G, \text{ad}^0(\mathfrak{p}))^r \) for the image of \( H^1(G, \text{ad}^0(\mathfrak{p})) \) in \( H^1(G, \text{ad}(\mathfrak{p})) \) under the map in cohomology coming from the inclusion of \( \text{ad}^0(\mathfrak{p}) \) in \( \text{ad}(\mathfrak{p}) \).

1. There are isomorphisms of \( k \)-vector spaces

\[
D^{\square,r}_{\mathfrak{p}}(k[\varepsilon]) \cong Z^1(G, \text{ad}(\mathfrak{p})) \quad \text{and} \quad D^{\square,r}_\mathfrak{p} \cong Z^1(G, \text{ad}^0(\mathfrak{p})).
\]

2. There are isomorphisms of \( k \)-vector spaces

\[
D_\mathfrak{p}(k[\varepsilon]) \cong H^1(G, \text{ad}(\mathfrak{p})) \quad \text{and} \quad D^\psi_{\mathfrak{p}}(k[\varepsilon]) \cong H^1(G, \text{ad}^0(\mathfrak{p}))^r.
\]

3. Assume \( H^2(G, \text{ad}^0(\mathfrak{p})) \) is finite dimensional. Set

- \( g = \dim_k Z^1(G, \text{ad}(\mathfrak{p})) \) and \( r = \dim_k H^2(G, \text{ad}(\mathfrak{p})) \),

- \( g_0 = \dim_k Z^1(G, \text{ad}^0(\mathfrak{p})) \) and \( r_0 = \dim_k H^2(G, \text{ad}^0(\mathfrak{p})) \).

There are presentations

\[
R^{\square,r}_{\mathfrak{p}}/\mathfrak{m}_\Lambda \cong k[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r) \quad \text{and} \quad R^{\square,r,\psi}_{\mathfrak{p}}/\mathfrak{m}_\Lambda \cong k[[y_1, \ldots, y_{g_0}]]/(h_1, \ldots, h_{r_0}).
\]

In particular, each irreducible component of \( \text{Spec}(R^{\square,r}_{\mathfrak{p}}/\mathfrak{m}_\Lambda) \) has dimension at least

\[
\dim_k Z^1(G, \text{ad}(\mathfrak{p})) - \dim_k H^2(G, \text{ad}(\mathfrak{p})),
\]

and each irreducible component of \( \text{Spec}(R^{\square,r,\psi}_{\mathfrak{p}}/\mathfrak{m}_\Lambda) \) has dimension at least

\[
\dim_k Z^1(G, \text{ad}^0(\mathfrak{p})) - \dim_k H^2(G, \text{ad}(\mathfrak{p})).
\]
4. Assume $D_{\varphi}$ is representable, and that $H^2(G, \text{ad}^0(\varphi))$ is finite dimensional. Set
\[
g = \dim_k H^1(G, \text{ad}(\varphi)) \quad \text{and} \quad r = \dim_k H^2(G, \text{ad}(\varphi))
\]
and $g_0 = \dim_k H^1(G, \text{ad}^0(\varphi))'$ and $r_0 = \dim_k H^2(G, \text{ad}^0(\varphi))$.

There are presentations
\[
R_{\varphi}/m_A \cong k[[x_1, \ldots, x_\rho]]/(f_1, \ldots, f_\rho) \quad \text{and} \quad R_{\varphi}/m_A \cong k[[y_1, \ldots, y_\rho]]/(h_1, \ldots, h_{r_0}).
\]
In particular, each irreducible component of Spec($R_{\varphi}/m_A$) has dimension at least
\[
\dim_k H^1(G, \text{ad}(\varphi)) - \dim_k H^2(G, \text{ad}(\varphi))
\]
and each irreducible component of Spec($R_{\varphi}/m_A$) has dimension at least
\[
\dim_k H^1(G, \text{ad}^0(\varphi))' - \dim_k H^2(G, \text{ad}^0(\varphi)).
\]

Proof. It is straightforward to check that the map $Z^1(G, \text{ad}(\varphi)) \rightarrow D_{\varphi}\left(k[\varepsilon]\right)$ given by $c \mapsto (1 + \varepsilon c)\varphi$ is an isomorphism that identifies $Z^1(G, \text{ad}^0(\varphi))$ with $D_{\varphi}\left(k[\varepsilon]\right) \subseteq D_{\varphi}\left(k[\varepsilon]\right)$. It is also straightforward to check that two cocycles $c$ and $c'$ yield the same deformation if and only if they differ by an $\text{ad}(\varphi)$-valued coboundary. This shows 1. and 2.

Parts 3. and 4. are proved as in [Maza, §1.6] (see also the proof of [Kis07, Lemma 4.1.1]). We give the proof that $R_{\varphi}/m_A \cong k[[y_1, \ldots, y_\rho]]/(h_1, \ldots, h_{r_0})$ with $g_0 = \dim_k Z^1(G, \text{ad}^0(\varphi))$ and $r_0 = \dim_k H^2(G, \text{ad}^0(\varphi))$.

The relative tangent space of $R_{\varphi}/m_A$ is isomorphic to
\[
\text{Hom}_{\text{C}_N}(R_{\varphi}^{\varphi}, k[\varepsilon]) = D_{\varphi}\left(k[\varepsilon]\right) \cong Z^1(G, \text{ad}^0(\varphi)),
\]
by part 1. Letting $A = k[[y_1, \ldots, y_\rho]]$, with $g_0 = \dim_k Z^1(G, \text{ad}^0(\varphi))$, we have a minimal presentation $R_{\varphi}/m_A \cong A/J$ for some ideal $J \subseteq m^2_A$, and it suffices to show $\dim_k J/m_A J \leq \dim_k H^2(G, \text{ad}^0(\varphi))$. For each $n \geq 2$, let $A_n = A/m^n_A$, $R_n = R_{\varphi}/m_A^n$, and $J_n = \ker(A_n \rightarrow R_n)$. For $m$ sufficiently large, the natural map $J/m_A J \rightarrow J_n/m_A J_n$ is an isomorphism of $k$-vector spaces, so it suffices to show that $\dim_k J_n/m_A J_n, k \leq \dim_k H^2(G, \text{ad}^0(\varphi))$. Let $\rho_n$ be the pushforward of the universal lift to $R_n$. Note that $\rho_n$ is universal for lifts of $\varphi$ to $k$-algebras $B$ in $\text{C}_N$ such that $m^2_B = 0$.

We consider the exact sequence
\[
0 \rightarrow J_n/m_A J_n \rightarrow A_n/m_A J_n \rightarrow R_n \rightarrow 0.
\]
Choose a continuous function $s : G \rightarrow \text{GL}_d(A_n)$ such that its composite with the surjection $\text{GL}_d(A_n) \rightarrow \text{GL}_d(R_n)$ is $\rho_n$, and such that $\det(s(g)) = \psi(g)$ for all $g \in G$. If $k$ is finite over $\mathbb{F}_p$, then $A_n$ and $R_n$ are finite, so it is obvious that such an $s$ exists. If $k$ is finite over $\mathbb{Q}_p$, then $A_n$ and $R_n$ are finite dimensional $k$-vector spaces, so any $k$-vector space map $R_n \rightarrow A_n$ will induce a continuous function $\text{GL}_d(R_n) \rightarrow \text{GL}_d(A_n)$, and precomposing this with $\rho_n$, we have a continuous function $\tilde{s} : G \rightarrow \text{GL}_d(A_n)$ that lifts $\rho_n$. We then let $s : G \rightarrow \text{GL}_d(A_n)$ be the continuous function obtained from $\tilde{s}$ by multiplying the first column of $\tilde{s}(g)$ by $\psi(g)(\det(\tilde{s}(g)))^{-1}$ for each $g \in G$, leaving the other columns alone, and this function $s$ has the desired properties.

We form the two cocycle $c(\sigma, \tau) = s(\sigma) s(\tau)^{-1} s(\sigma)^{-1}$ on $G$, which takes values in $\text{ad}^0(\varphi) \otimes_k J_n/m_A J_n$, and we let $O(\rho)$ denote the class of this cocycle in
\[
H^2(G, \text{ad}^0(\varphi) \otimes_k J_n/m_A J_n) \cong H^2(G, \text{ad}^0(\varphi)) \otimes_k J_n/m_A J_n.
\]
We then have a map $\text{Hom}_k(J_n/m_A J_n, k) \rightarrow H^2(G, \text{ad}^0(\varphi))$ given by $\lambda \mapsto (1 \otimes \lambda)(O(\rho))$. Given a nonzero $\lambda \in \text{Hom}_k(J_n/m_A J_n, k)$, we can push (1) forward via $\lambda$ to obtain an exact sequence
\[
0 \rightarrow k \rightarrow A' \rightarrow R_n \rightarrow 0,
\]
such that $A' \rightarrow R_n$ still induces an isomorphism on tangent spaces. If $(1 \otimes \lambda)(O(\rho)) = 0$, then the 2-cocycle $G \times G \rightarrow \text{ad}^0(\varphi) \otimes_k J_n/m_A J_n \xrightarrow{1 \otimes \lambda} \text{ad}^0(\varphi)$ is a coboundary, and we can modify the map $G \rightarrow \text{GL}_d(A_n/m_A J_n) \rightarrow \text{GL}_d(A')$ by this coboundary so that it becomes a homomorphism, hence a lift of $\varphi$ with determinant $\psi$. By the universal property of $\rho_n$, this would give a $k$-algebra section of (2), contradicting the fact that $A' \rightarrow R_n$ induces a surjection on tangent spaces. This shows that the map $\text{Hom}_k(J_n/m_A J_n, k) \rightarrow H^2(G, \text{ad}^0(\varphi))$ is injective, so $\dim_k J_n/m_A J_n \leq \dim_k H^2(G, \text{ad}^0(\varphi))$. \qed
In what follows, we will need to compare the fixed determinant ring with the non-fixed determinant ring. When \( n \) is invertible in \( k \), this is usually done by tensoring the fixed determinant ring with the \( n \)th root of the universal character for the (one dimensional) trivial representation. We will prove a fairly general result below for finite \( k \) that holds even when \( pd \). It is inspired by (and uses) many of the arguments in [KW09].

1.1.10. We now specialize to the case when \( \Lambda \) is the ring of integers of a finite extension of \( \mathbb{Q}_p \), and we change notation, writing \( \mathcal{O} \) for \( \Lambda \) and \( \mathbb{F} \) for \( k \). Let \( T \) be a finite set and if \( T = \emptyset \) assume that \( \mathrm{End}_{\mathcal{O}[G]}(V_\tau) = \mathbb{F} \). Let \( R \) be a deformation quotient of \( R^{\mathbb{Q}_p} \) and let \( \psi : G \to \mathbb{O}^\times \) be a continuous character such that \( \det(\psi) = \psi \mod m_\mathcal{O} \). We let \( R^{\psi} \) denote the further deformation quotient that corresponds to the subfunctor of \( \mathrm{Spf}R \) of \( T \)-framed deformations with determinant \( \psi \). Choose any lift \( \rho^{\mathbb{Q}_p} : G \to \mathrm{GL}_d(R^{\mathbb{Q}_p}) \) in the universal \( T \)-framed deformation class, and let \( \rho : G \to \mathrm{GL}_d(R) \) denote the pushforward of \( \rho^{\mathbb{Q}_p} \) via the surjection \( R^{\mathbb{Q}_p} \to R \). Let \( \Gamma \) be a pro-\( p \) abelian quotient of \( G \). We assume \( R \) and \( \Gamma \) satisfying the following:

(a) \( \psi^{-1} \det(\rho_R) : G \to R^\times \) factors through \( \Gamma \);
(b) for any \( \mathbb{CNL}_D \)-algebra \( A \), any \( [(\rho, (\alpha_v))] \in D_\Gamma(A) \), and any continuous character \( \chi : G \to 1 + m_\mathcal{O} \) factoring through \( \Gamma \), the \( T \)-framed deformation \( [\rho, (\alpha_v)] \) belongs to \( \mathrm{Spf}R(A) \subseteq D_\Gamma(A) \) if and only if \( [\rho \otimes \chi, (\alpha_v)] \) does.

Condition (a) implies that \( \psi^{-1} \det(\rho_R) \) induces a \( \mathcal{O}[\Gamma] \)-algebra morphism \( \mathcal{O}[\Gamma] \to R \), and \( R^{\psi} \) is equal to the quotient of \( R \) by the augmentation ideal of \( \mathcal{O}[\Gamma] \) under this map.

Note that \( \mathcal{O}[\Gamma] \) represents the set valued funtor on \( \mathbb{CNL}_D \) that sends a \( \mathbb{CNL}_D \)-algebra \( A \) to the set of continuous characters \( \chi : G \to 1 + m_\mathcal{O} \), so \( \mathcal{O}[\Gamma] \) is functor on \( \mathbb{CNL}_D \) valued in abelian groups, and we denote it by \( \mathcal{G}(\Gamma) \). Since \( \mathcal{G}(\Gamma) \) is a finitely generated \( \mathbb{Z}_p \)-module, it is easy to see that \( \mathcal{G}(\Gamma) \) is diagonalizable group in the \( \mathbb{CNL}_D \) category in the sense of [KW09, §2.5] (so we may apply [KW09, Proposition 2.6] below).

Condition (b) implies that we have an action of \( \mathcal{G}(\Gamma) \) on \( \mathrm{Spf}R \) by twists, i.e. on \( A \)-points \( \mathcal{G}(\Gamma)(A) \times \mathrm{Spf}R(A) \to \mathrm{Spf}R(A) \) is given by \( (\chi, [\rho, (\alpha_v)]) \mapsto [\rho \otimes \chi, (\alpha_v)] \). We let \( R^{\mathbb{inv}} \) be the subalgebra of \( R \) of elements invariant under \( \mathcal{G}(\Gamma) \). More precisely, if \( \gamma : R \to \mathcal{O}[\Gamma], \mathcal{O} \to \mathcal{O} \) be the \( \mathbb{CNL}_D \)-algebra map corresponding to \( \mathcal{G}(\Gamma) \times \mathrm{Spf}R \to \mathrm{Spf}R \), then \( R^{\mathbb{inv}} = \{ r \in R \mid \gamma(r) = 1 \otimes r \} \).

Finally, let \( \delta : \mathrm{Spf}R \to \mathcal{G}(\Gamma) \) denote the map in \( \mathcal{CNL}_D \) induced by \( \psi^{-1} \det(\rho_R) \). Note that \( \mathcal{G}(\Gamma) \) is identified with the fibre \( \delta = 1 \), and that for any \( \mathbb{CNL}_D \)-algebra \( A \), and points \( g \in \mathcal{G}(\Gamma)(A) \) and \( x \in \mathrm{Spf}R(A) \), we have \( \delta(g, x) = g^d \delta(x) \).

Proposition 1.1.11. Let the notation and assumptions be as above 1.1.10. Assume also that the action of \( \mathcal{G}(\Gamma) \) on \( \mathrm{Spf}R \) is free, i.e. that the map \( \mathcal{G}(\Gamma) \times \mathrm{Spf}R \to \mathrm{Spf}R \times \mathrm{Spf}R \) given on points by \( (g, x) \mapsto (x, gx) \) is a closed embedding. Then the following hold.

1. A quotient \( \mathcal{G}(\Gamma) \backslash \mathrm{Spf}R \) exists in \( \mathbb{CNL}_D^{\text{op}} \) and is represented by \( R^{\mathbb{inv}} \), i.e. \( R^{\mathbb{inv}} \) is a \( \mathcal{CNL}_D \)-algebra and \( \mathrm{Spf}R \to \mathrm{Spf}R^{\mathbb{inv}} \) is universal for \( \mathcal{G}(\Gamma) \)-morphisms \( \mathrm{Spf}R \to Y \) in \( \mathbb{CNL}_D^{\text{op}} \) with \( \mathcal{G}(\Gamma) \) acting trivially on \( Y \).
2. The inclusion \( R^{\mathbb{inv}} \to R \) is faithfully flat and \( \mathrm{Spf}R \) is a torsor on \( \mathrm{Spf}R^{\mathbb{inv}} \) for \( \mathcal{G}^{\mathbb{inv}} \) (in particular, \( \dim R^{\mathbb{inv}} = \dim R - \text{rank}_{\mathbb{Z}_p} \Gamma \)). The inclusion \( R^{\mathbb{inv}}[1/p] \to R[1/p] \) is formally smooth. If \( \Gamma \) is torsion free, then \( R^{\mathbb{inv}} \to R \) is formally smooth.
3. Let \( \mathcal{G}(\Gamma)[d] \) be the finite \( \mathcal{O} \)-group scheme of \( d \)-torsion in \( \mathcal{G}(\Gamma) \). The map \( R^{\mathbb{inv}} \to R^{\psi} \) makes \( \mathrm{Spf}R^{\psi} \) a torsor on \( \mathrm{Spf}R^{\mathbb{inv}} \) for \( \mathcal{G}(\Gamma)[d]^{\mathbb{inv}} \). In particular, \( \dim R^{\psi} = \dim R - \text{rank}_{\mathbb{Z}_p} \Gamma \) and \( \text{and } R^{\mathbb{inv}}[1/p] \to R^{\psi}[1/p] \) is finite étale.

Proof. Part 1. and the first sentence of 2. is [KW09, Proposition 2.6(1)] (the faithful flatness is not explicitly mentioned but follows from the proof since \( \mathcal{O}[\Gamma] \) is faithfully flat over \( \mathcal{O} \)). If \( \Gamma \) is torsion free, then the smoothness of \( R^{\mathbb{inv}} \to R \) is part of [KW09, Proposition 2.5]. We now show that \( R^{\mathbb{inv}}[1/p] \to R[1/p] \) is formally smooth in general. Let \( \Delta \) be the torsion subgroup of \( \Gamma \). Then \( \mathcal{G}(\Gamma) / \Delta = \mathcal{Spf}(\mathcal{O}[\Gamma] / \Delta) \) is a closed \( \mathcal{CNL}_D \)-subgroup of \( \mathcal{G}(\Gamma) \), and the quotient is naturally isomorphic to \( \mathcal{G}(\Delta) = \mathcal{Spf}(\mathcal{O}[\Delta]) \), which is finite over \( \mathcal{O} \). Let \( R' \) denote the subring of \( R \) of elements invariant under \( \mathcal{G}(\Gamma) / \Delta \); we have inclusions \( R^{\mathbb{inv}} \to R' \to R \). Since \( \Gamma / \Delta \) is torsion free, \( R' \to R \) is formally smooth (again by [KW09, Proposition 2.5]), and we are reduced to showing \( R^{\mathbb{inv}}[1/p] \to R'[1/p] \) is formally smooth. By [KW09, Proposition 2.6(2)], \( R^{\mathbb{inv}} \) represents the quotient of \( \mathrm{Spf}R' \) by the induced free action of \( \mathcal{G}(\Delta) \), and \( \mathrm{Spf}R' \to \mathrm{Spf}R^{\mathbb{inv}} \) is a torsor for the finite \( R^{\mathbb{inv}} \)-group \( \mathcal{G}(\Delta)^{\mathbb{inv}} \), which becomes trivial under the finite faithfully flat cover \( \mathrm{Spf}R' \to \mathrm{Spf}R^{\mathbb{inv}} \). Using
this, the map 
\[
R'[1/p] \to R'[1/p] \otimes_{R^\mathrm{inv}[1/p]} R'[1/p] = (R' \otimes_{R^\mathrm{inv}} R')[1/p] \\
\cong (R' \otimes_{R^\mathrm{inv}} R^\mathrm{inv}([\Delta])[1/p]) = R'[1/p] \otimes_{R^\mathrm{inv}[1/p]} R^\mathrm{inv}[\Delta][1/p]
\]
is finite étale. Since this is the base change of \(R^\mathrm{inv}[1/p] \to R'[1/p]\) via the finite faithfully flat map \(R^\mathrm{inv}[1/p] \to R'[1/p]\), we deduce that \(R^\mathrm{inv}[1/p] \to R'[1/p]\) is also finite étale (see [Gro67, Corollary 17.7.3]), which completes the proof of 2.

It remains to prove 3, and we proceed exactly as in [KW09, Lemma 9.4]. Consider the fibre product
\[
\begin{array}{ccc}
\text{Spf} R \times_{G(\Gamma)} G(\Gamma) & \to & G(\Gamma) \\
\downarrow & & \downarrow d \\
\text{Spf} R & \to & G(\Gamma)
\end{array}
\]
Recall that on points this corresponds to pairs \([\rho, (\alpha_v)]\), \(\chi\), where \([\rho, (\alpha_v)]\) is a \(T\)-framed deformation and \(\chi\) is a character, such that \(\det(\rho) = \chi^d\). Define an action of \(G(\Gamma) \times G(\Gamma)[d]\) on \(\text{Spf} R \times_{G(\Gamma)} G(\Gamma)\) by letting \(G(\Gamma)\) act diagonally, and letting \(G(\Gamma)[d]\) act trivially on the first factor and by translations on the second factor. Using the fact that \(\text{Spf} R \to \text{Spf} R^\mathrm{inv}\) is a \((G(\Gamma) \otimes_{G(\Gamma)} G(\Gamma))\)-torsor, it is straightforward to check that \(\text{Spf} R \times_{G(\Gamma)} G(\Gamma) \to \text{Spf} R^\mathrm{inv}\) is a \((G(\Gamma) \times G(\Gamma)[d])\)-torsor. There is an isomorphism \(\text{Spf} R \times_{G(\Gamma)} G(\Gamma) \sim \text{Spf} R^\psi \times G(\Gamma)\) given on points by \((x, g) \mapsto (g^{-1}x, g)\). Under this isomorphism \(G(\Gamma)\) acts trivially on \(\text{Spf} R^\psi\) and by translation on itself. This action is clearly free, so [KW09, Proposition 2.6] gives a free action of \(G(\Gamma)[d]\) on \(\text{Spf} R^\psi\) whose quotient is \(\text{Spf} R^\mathrm{inv}\), and \(\text{Spf} R^\psi \to \text{Spf} R^\mathrm{inv}\) is a \(G(\Gamma)[d]\)-torsor. Finally, arguing as in the proof of 2., we see that \(R^\mathrm{inv}[1/p] \to R^\psi[1/p]\) is finite étale.

The following lemma gives two typical situations when the freeness of the action in 1.1.11 is satisfied.

**Lemma 1.1.12.** Let \(\Gamma\) and \(R\) be as above. The action \(G(\Gamma) \times \text{Spf} R \to \text{Spf} R\) is free in either of the following situations.

1. \(T \neq \emptyset\).
2. \(\overline{p}|_H\) is absolutely irreducible for any index \(p\) open subgroup \(H\) of \(G\).

**Proof.** A morphism \(X \to Y\) in \(\text{CNL}_{O}^p\) is a closed embedding if and only if it is a monomorphism, so it suffices to show the action is free on points.

Let \(A\) be a \(\text{CNL}_{O}\)-algebra, and let \([\rho, (\alpha_v)] \in \text{Spf} R(A)\) and \(\chi \in G(\Gamma)(A)\). If \([\rho, (\alpha_v)] = [\rho \otimes \chi, (\alpha_v)]\), then there is \(g \in \ker(\text{GL}_d(A) \to \text{GL}_d(F))\) such that \(gpg^{-1} = \rho \otimes \chi\) and \(g \alpha_v = \alpha_v\) for each \(v \in T\). If \(T \neq \emptyset\), then \(g = 1\) and \(p = \rho \otimes \chi\), hence \(\chi = 1\). This shows 1.

To show 2., we assume that \(\overline{p}\) is absolutely irreducible and show that if \([\rho] = [\rho \otimes \chi]\) with \(\chi \neq 1\), then there is an index \(p\) open subgroup \(H\) of \(G\) such that \(\overline{p}|_H\) becomes absolutely reducible. It suffices to show this for points valued in finite length elements \(A\) of \(\text{CNL}_{O}\). Let \(N = \ker(\chi)\). Since \(A\) is finite length over \(O\), we see that \(N\) is an open subgroup of index a power of \(p\). Since \(\chi \neq 1\), the element \(g\) above satisfying \(gpg^{-1} = \rho \otimes \chi\) is not scalar, so the centralizer of the image of \(\rho|_N\) is nonscalar. This implies \(\overline{p}|_N\) is not absolutely irreducible (see, for instance, [Gou, Lemma 9.4 of Appendix 1]). By Clifford theory, there is a continuous representation \(\varrho : G \to \text{GL}_m(F)\), with \(m = d/[G : N]\), such that \(\overline{p} \cong \text{Ind}_{\varrho}^{G}(\varrho)\). We can then take \(H\) to be any index \(p\) subgroup of \(G\) containing \(N\).

Even when \(T = \emptyset\) and \(\overline{p}\) does not satisfy 2. of 1.1.11, we can still deduce information on the relation between the fixed and non-fixed determinant deformation rings by applying by using the framed deformation rings and 1.1.7. We state one example.

**Corollary 1.1.13.** Assume that \(\text{End}_{F|G}(V_\varrho) = F\). Let \(G^\text{ab}(p)\) denote the maximal pro-\(p\) quotient of the abelianization of \(G\). Then

1. \(\dim R_\varrho = \dim R^\psi_\varrho + \rank_{Z_p} G^\text{ab}(p)\).
2. \(R_\varrho\) is \(O\)-flat if and only if \(R^\psi_\varrho\) is \(O\)-flat.
3. Let \(x\) be a closed point of \(\text{Spec} R_\varrho^\psi[1/p] \subseteq \text{Spec} R_\varrho[1/p]\). Then \(x\) is a smooth point of \(\text{Spec} R_{\varrho}^\psi[1/p]\) if and only if it is a smooth point of \(\text{Spec} R_{\varrho}[1/p]\).
1.2. Weil–Deligne representations. Let \( \ell \) be a rational prime, and let \( K \) be a finite extension of \( \mathbb{Q}_\ell \) with ring of integers \( \mathcal{O}_K \) and uniformizer \( \varpi_K \). Let \( q \) denote the cardinality of the residue field of \( K \), and let \( |\cdot| \) denote the absolute value on \( K \) normalized so that \( |\varpi_K| = q^{-1} \). We fix an algebraic closure \( \overline{K} \) of \( K \) and set \( G_K = \text{Gal}(\overline{K}/K) \). We let \( I_K \) denote the inertia subgroup of \( G_K \), and let \( W_K \) denote the Weil group of \( K \). Recall that the Artin reciprocity map \( \text{Art} : K^\times \to W_K^{ab} \) is normalized so that uniformizers correspond to geometric Fobenii. For \( w \in W_K \), we will write \( |w| \) for \( |\text{Art}^{-1}(w)| \).

1.2.1. We recall some basics of Weil–Deligne representations (see [Tat77, §4]). Given a characteristic 0 field \( \Omega \), a Weil–Deligne representation over \( \Omega \) is a pair \((r,N)\), where \( r : W_K \to \text{Aut}_\Omega(V) \) is a representation of \( W_K \) on a finite dimensional \( \Omega \)-vector space \( V \) with open kernel, and \( N \in \text{End}_\Omega(V) \) is nilpotent, such that \( r(w)N(r(w)^{-1}) = |w|N \) for all \( w \in W_K \). A morphism of Weil–Deligne representations \((r_1,N_1) \to (r_2,N_2)\) is an \( \Omega \)-linear morphism that intertwines the \( r_i \) and the \( N_i \). A Weil–Deligne representation \((r,N)\) is called Frobenius-semisimple if \( r \) is semisimple (equivalently, if \( r(\Phi) \) is semisimple for \( \Phi \in W_K \) a lift of the Frobenius).

Given a Weil–Deligne representation \((r,N)\), we will denote by \((r,N)^{F-ss}\) its Frobenius-semisimplification, i.e. \((r,N)^{F-ss} = (r^{ss},N)\). Given a Weil–Deligne representation \((r,N)\), we denote by \((r(1),N)\) the Weil–Deligne representation given by \( r(1)(w) = |w|r(w) \). If \( \iota \in \text{Aut}(\Omega) \), we let \( \iota(r,N) = (r,\iota N) \) denote the Weil Deligne representation obtained by change of scalars via \( \iota : \Omega \to \Omega \) (this is again a Weil–Deligne representation since \( |w| \in \Omega \) for all \( w \in W_K \)).

**Definition 1.2.2.** We say a Weil–Deligne representation \((r,N)\) is generic if there is no nontrivial morphism \((r,N) \to (r(1),N)\).

If \( \pi \) is an irreducible admissible representation of \( \text{GL}_d(K) \) over \( \mathbb{C} \) and \( \iota \in \text{Aut}(\mathbb{C}) \), then \( \text{rec}_K^\iota(\iota \pi) = \iota \text{rec}_K^\iota(\pi) \) (this is explained when \( d = 2 \) in [BH06, §35] and the argument there generalizes using [BH00, Theorem 3.2] and the converse theorems of [Hen93]). If \( \Omega \) is algebraically closed with cardinality continuum, we get a bijection, again denoted by \( \text{rec}_K^\iota \), between isomorphism classes of irreducible admissible representations of \( \text{GL}_d(K) \) over \( \Omega \) and isomorphism classes of \( d \)-dimensional Frobenius-semisimple Weil–Deligne representations over \( \Omega \) by fixing any isomorphism \( \iota : \mathbb{C} \to \Omega \) and setting \( \text{rec}_K^\iota(\pi) = \iota \text{rec}_K^\iota((\iota^{-1} \pi)) \), and this is independent of the choice of \( \iota \).

**Lemma 1.2.3.** Let \( \pi \) be an irreducible smooth admissible representation of \( \text{GL}_d(K) \) on an \( \Omega \)-vector space, with \( \Omega \) algebraically closed field of characteristic 0 and cardinality continuum. Then \( \text{rec}_K^\iota(\pi) \) is generic if and only if \( \pi \) is generic.

**Proof.** This is essentially identical to [BLGGT14, Lemma 1.3.2(1)]. We give the details. It suffices to consider the case \( \Omega = \mathbb{C} \). Since \( \pi \) is generic if and only if \( \pi \otimes |\cdot|^\frac{1}{d} \) is generic, it is equivalent to show that \( \pi \) is generic if and only if \( \text{rec}_K(\pi) \) is generic. Note that if \( (r,N) = \text{rec}_K(\pi) \), then \( (r(1),N) = \text{rec}_K(\pi \otimes |\cdot|) \).

We will use the notation and terminology of [HT01, §1.3]. There are positive integers \( s_i,d_i \) for \( i = 1,\ldots,t \) with \( d = d_1s_1 + \cdots + d_ts_t \) and irreducible supercuspidal representations \( \pi_i \) of \( \text{GL}_d(K) \) such that
\[
\pi \cong \text{Sp}_{s_1}(\pi_1) \boxplus \cdots \boxplus \text{Sp}_{s_t}(\pi_t),
\]
and the multiset \( \{(s_1,\pi_1),\ldots,(s_t,\pi_t)\} \) is uniquely determined by \( \pi \). By abuse of notation, we also denote by \( \text{Sp}_d \) the \( k \)-dimensional Weil–Deligne representation \((r,N)\) on a complex vector space with basis \( e_0,\ldots,e_{k-1} \), where \( r(w) = |w|e_i \) for each \( i = 0,\ldots,n-1 \), and \( N e_{i} = e_{i+1} \) for each \( i = 0,\ldots,k-2 \) and \( N e_{k-1} = 0 \). Then (see [HT01, Theorem 7.2.20] and the discussion preceding it)
\[
\text{rec}_K(\pi) = (\text{rec}_K(\pi_1) \otimes \text{Sp}_{s_1}) \oplus \cdots \oplus (\text{rec}_K(\pi_t) \otimes \text{Sp}_{s_t}),
\]
and \( \text{rec}_K(\pi) \) is nongeneric if and only if
\[
(3) \quad \text{Hom}_{\text{WD}}(\text{rec}_K(\pi_1) \otimes \text{Sp}_{s_1},\text{rec}_K(\pi_2 \otimes |\cdot|) \otimes \text{Sp}_{s_1}) \neq 0.
\]
for some $i, j$. Since $\rec_K(\pi_i)$ is absolutely irreducible (as $\pi_i$ is supercuspidal), it is easy to check that (3) holds if and only if $\pi_i \cong \pi_i \otimes |\cdot|^a$ with $s_j - s_i < a \leq s_j$. In the notation and terminology [Zel80] (see [Zel80, §3.1 and §4.1]), this happens if and only if the segments $[\pi_i, \ldots, \pi_i \otimes |\cdot|^u]$ and $[\pi_j, \ldots, \pi_j \otimes |\cdot|^u]$ are linked, which happens if and only if $\pi$ is non-generic by [Zel80, Theorem 9.7] (note generic is called non-degenerate in [Zel80]).

1.2.4. Assume that $\ell \neq p$, and let

$$\rho : G_K \rightarrow \GL_d(E)$$

be a continuous representation. Following [Tat77, §4.2], we can attach a Weil–Deligne representation to our fixed $\rho$, that we will denote $\WD(\rho)$, as follows. Fix $\Phi \in G_K$ mapping to the geometric Frobenius in $G_K/I_K$. Fix a surjection $t_p : I_K \rightarrow \Z_p$, and let $\tau_p \in I_K$ be such that $t_p(\tau_p) = 1$. The homomorphism $t_p$ necessarily factors through tame inertia. Write $\rho(\tau_p) = \rho(\tau_p)^u = \rho(\tau_p)^u$ with $\rho(\tau_p)^u$ semisimple and $\rho(\tau_p)^u$ unipotent. Set $N = \log(\rho(\tau_p)^u)$. Then the map $r : W_F \rightarrow \Aut_E(V_{\rho})$ given by

$$r(\Phi^\sigma) = \rho(\Phi^\sigma)e^{-t_p(\sigma)N}$$

for $n \in \Z$ and $\sigma \in I_K$, is well-defined with open kernel, and $(r, N)$ is a Weil–Deligne representation. The isomorphism class does not depend on the choices made, and we denote any element in this isomorphism class by $\WD(\rho)$. Moreover, this assignment (which depends on $\Phi$ and $t_p$) gives an equivalence of categories from the category of continuous representations $\rho : G_K \rightarrow \GL_d(E)$ to the full subcategory of Weil–Deligne representations $(r, N)$ on $E^d$ such that $r$ has bounded image. From this we deduce the following lemma.

**Lemma 1.2.5.** Let $\rho : G_K \rightarrow \GL_d(E)$ be a continuous representation. The Weil–Deligne representation $\WD(\rho)$ is generic if and only if $\Hom_{E[G_K]}(V_{\rho}, V_{\rho}(1)) = 0$.

1.2.6. Assume $\ell = p$, and let

$$\rho : G_K \rightarrow \GL_d(E)$$

be a continuous potentially semistable representation. Following Fontaine, [Fon94b, §1.3 and §2.3], we can also associate a Weil–Deligne representation to $\rho$, again denoted $\WD(\rho)$, as follows.

Let $L/K$ be a finite extension. Let $G_{L/K} = \Gal(L/K)$, and let $L_0$ be the maximal subfield of $L$ unramified over $\Q_p$. We assume that $E$ contains all embeddings of $L_0$ into an algebraic closure of $E$. A $(\varphi, N, G_{L/K})$-module $D$ over $E$ is a finite free $L_0 \otimes_{\Q_p} E$-module together with operators $\varphi$ and $N$, and an action of $\Gal(L/K)$, satisfying the following:

- $N$ is $L_0 \otimes_{\Q_p} E$-linear;
- $\varphi$ is $E$-linear and satisfies $\varphi(ax) = \sigma(a)\varphi(x)$ for any $x \in D$ and $a \in L_0$, where $\sigma \in \Gal(L_0/\Q_p)$ is the absolute arithmetic Frobenius;
- $N_\varphi = p\varphi N$;
- the $\Gal(L/K)$-action is $E$-linear and $L_0$-semilinear, and commutes with $\varphi$ and $N$.

Extend the action of $\Gal(L/K)$ to $W_K$ by letting $I_L$ act trivially. For $w \in W_K$, we let $v(w) \in \Z$ be such that the image of $w$ in $W_K/I_K$ is $\sigma^{-v(w)}$. We then define an $L_0 \otimes_{\Q_p} E$-linear action, denoted $\tau_D$, of $W_K$ on $D$ by $\tau_D(w) = w\varphi^{v(w)}$. Writing $L_0 \otimes_{\Q_p} E = \prod_{\tau : L_0 \rightarrow E} D_\tau$, we get a decomposition $D = \prod_{\tau : L_0 \rightarrow E} D_\tau$ and an induced $n$-dimensional Weil–Deligne representation $(r_\tau, N_\tau)$ over $E$ on each factor $D_\tau$. The isomorphism class of $(r_\tau, N_\tau)$ is independent of $\tau : L_0 \rightarrow E$ (see [BM02, §2.2.1]), and we denote any element in its isomorphism class by $\WD(D)$. Moreover, by [BS07, Proposition 4.1], this assignment induces an equivalence of categories from $(\varphi, N, G_{L/K})$-modules over $E$ to Weil–Deligne representations over $E$ on which $I_L$ acts trivially. Given a $(\varphi, N, G_{L/K})$-module $D$ over $E$, we let $D(1)$ be the $(\varphi, N, G_{L/K})$-module the same underlying $L_0 \otimes_{\Q_p} E$-module, operator $N$, and $G_{L/K}$-action, but with $\varphi(D) = p^{-1}\varphi D$. Note that $\WD(D(1)) = \WD(D)(1)$.

Now choose $L/K$ such that $\rho|_{G_L}$ is semistable and such that $E$ contains all embeddings of $L$ in $\Q_p$. Letting $B_{st}$ denote Fontaine’s ring of semistable periods, we have a $(\varphi, N, G_{L/K})$-module

$$D_{st,L}(\rho) := (B_{st} \otimes_{\Q_p} V_{\rho})^{G_L}.$$

The isomorphism class of $\WD(D_{st,L}(\rho))$ does not depend on the choice of $L$ (see [BM02, §2.2.1]), and we set $\WD(\rho) = \WD(D_{st,L}(\rho))$.

**Lemma 1.2.7.** Let $\rho : G_K \rightarrow \GL_d(E)$ be a potentially semistable representation. Let $L/K$ be a finite representation such that $\rho|_{G_L}$ is semistable.
1. The Weil–Deligne representation \( \text{WD}(\rho) \) is generic if and only if the only morphism \( D_{st,L}(\rho) \to D_{st,L}(\rho)(1) \) of \((\varphi, N, G_{L/K})\)-modules over \( E \) is the trivial one.

2. Let \( \text{ad}(\rho) = M_{d \times d}(E) \) with the adjoint \( \text{Gr} \)-action. Let \( B_{ct} \) be Fontaine’s ring of crystalline periods, and let \( D_{ct}(\text{ad}(\rho)(1)) = (B_{ct} \otimes_{\mathbb{Q}_p} \text{ad}(\rho)(1))^{G_K} \) with its induced crystalline Frobenius \( \varphi \). Then \( \text{WD}(\rho) \) is generic if and only if \( D_{ct}(\text{ad}(\rho)(1))^{\varphi=1} = 0 \).

3. If \( \text{WD}(\rho) \) is generic, then \( \text{Hom}_{E[G_K]}(V_{\rho}, V_{\rho}(1)) = 0 \).

**Proof.** Part 1. follows from [BS07, Proposition 4.1]. Part 3. follows from part 1. and the fact that \( D_{st,L}(\rho(1)) = D_{st,L}(\rho) \otimes_{L_0} D_{st,L}(\epsilon) = D_{st,L}(\rho)(1) \) (see [Fon94a, Théorème 5.1]), where we view \( \epsilon \) as a one-dimensional \( \mathbb{Q}_p \)-representation of \( G_K \). We now prove 2.

By [Fon94a, §5.6], the \((\varphi, N, G_{L/K})\)-module \( D_{st,L}(\text{Hom}_{\mathbb{Q}_p}(V_{\rho}, V_{\rho}(1))) \) over \( \mathbb{Q}_p \) is identified with the \((\varphi, N, G_{L/K})\)-module over \( \mathbb{Q}_p \) consisting of \( L_0 \)-vector space of morphisms \( D_{st,L}(\rho) \to D_{st,L}(\rho(1)) \) with \((\varphi, N, G_{L/K})\)-module structure by

\[
\begin{align*}
\varphi f &= \varphi \circ f \circ \varphi^{-1}, \\
N f &= N \circ f - f \circ N, \\
\gamma f &= \gamma \circ f \circ \gamma^{-1}, \text{ for } \gamma \in G_{L/K}.
\end{align*}
\]

This identification takes the subspace of elements that commute with \( E \) to the subspace of elements that commute with \( E \), and we have an isomorphism \( D_{st,L}(\text{ad}(\rho)(1)) \) with the \((\varphi, N, G_{L/K})\)-module structure as above. This together with part 1. implies that \( \text{WD}(\rho) \) is generic if and only if

\[
\{ f \in D_{st,L}(\text{ad}(\rho)(1))^{G_{L/K}} \mid N f = 0 \text{ and } \varphi f = f \} = 0.
\]

The left hand side of this expression is exactly the subspace of \( D_{ct}(\text{ad}(\rho)(1)) \) on which \( \varphi = 1 \). □

The converse of part 3. of 1.2.7 is not true. For example, if \( \rho \) is a nonsplit crystalline extension of the trivial character by the cyclotomic character, then \( \text{Hom}_{E[G_K]}(V_{\rho}, V_{\rho}(1)) = 0 \), but \( \text{WD}(\rho) = (|\cdot| \pm 1, 0) \), which is nongeneric.

### 1.3. Local Galois deformation rings

Fix a continuous representation

\[
\overline{\rho} : G_K \longrightarrow \text{GL}_d(\mathbb{F}),
\]

and a continuous character \( \psi : G_K \to \mathcal{O}^\times \) such that \( \psi \mod m_{\mathcal{O}} = \det(\overline{\rho}) \). We let \( R_{\overline{\rho}}^{\square} \) be the universal lifting ring for \( \overline{\rho} \) (see 1.1.2) and \( R_{\overline{\rho}}^{\square,\psi} \) be the universal determinant \( \psi \) lifting ring for \( \overline{\rho} \) (see 1.1.8). We let \( \rho^{\square} : G_K \to \text{GL}_d(R_{\overline{\rho}}^{\square}) \) denote the universal lift.

In what follows, if \( R \) is a quotient of \( R_{\overline{\rho}}^{\square} \), and \( x \in \text{Spec} R[1/p] \) has residue field \( k \), we will again denote by \( x \) the \( E \)-algebra morphism \( x : R[1/p] \to k \). If \( x : R[1/p] \to A \) is any \( E \)-algebra morphism, we will denote by \( \rho_x : G_K \to \text{GL}_d(A) \) the specialization of \( \rho^{\square} \) via \( R^{\square} \to R[1/p] \xrightarrow{\sim} A \).

**Proposition 1.3.1.** Assume \( \ell \neq p \).

1. \( \text{Spec} R_{\overline{\rho}}^{\square}(1/p) \) is equidimensional of dimension \( d^2 \) and \( \text{Spec} R_{\overline{\rho}}^{\square,\psi}(1/p) \) is equidimensional of dimension \( d^2 - 1 \).

2. A closed point \( x \) of \( \text{Spec} R_{\overline{\rho}}^{\square}(1/p) \), resp. of \( \text{Spec} R_{\overline{\rho}}^{\square,\psi}(1/p) \), is smooth if and only if \( \text{WD}(\rho_x) \) is generic.

**Proof.** Letting \( \Gamma \) be the maximal pro-\( p \) quotient of \( G_K \), we see that \( R_{\overline{\rho}}^{\square} \) and \( \Gamma \) trivially satisfy assumptions (a) and (b) of 1.1.10. The action of the CNL\( \mathcal{O} \)-group \( G(\Gamma) \) on \( \text{Spf} R_{\overline{\rho}}^{\square} \) is free by 1.1.11. We can then use 1.1.11 to deduce the statements for \( R_{\overline{\rho}}^{\square,\psi} \) from those of \( R_{\overline{\rho}}^{\square} \) (note rank\( Z_{\overline{\rho}^\square} \Gamma = 1 \)).

The fact that \( \text{Spec} R_{\overline{\rho}}^{\square}(1/p) \) has dimension \( d^2 \) follows from [Gee11, Theorem 2.1.6] (see also the discussion preceding Proposition 2.1.4 of [Gee11]). Let \( k \) denote the residue field of \( x \). Then \( x \) is a smooth point if and only if the tangent space of \( (R_{\overline{\rho}}^{\square})_x \) has \( k \)-dimension \( d^2 \). By 1.1.5 and part 1. of 1.1.9, this is if and only if \( \dim_k Z^1(G_K, \text{ad}(\rho_x)) = d^2 \). By local Euler characteristic,

\[
\dim_k Z^1(G_K, \text{ad}(\rho_x)) = \dim_k H^1(G_K, \text{ad}(\rho_x)) + d^2 - \dim_k H^0(G_K, \text{ad}(\rho_x)) = d^2 + \dim_k H^2(G_K, \text{ad}(\rho_x)),
\]
so $(R^\psi_p)_x^\omega$ is formally smooth over $k$ if and only if $H^2(G_K, \text{ad}(\rho_x)) = 0$. The trace pairing on $\text{ad}(\rho_x)$ is perfect, so Tate local duality implies that $H^2(G_K, \text{ad}(\rho_x)) = 0$ if and only if $H^0(G_K, \text{ad}(\rho_x)(1)) = 0$. This is equivalent to $\text{Hom}_{k[G_K]}(V_{\rho_x}, V_{\rho_x}(1)) = 0$, which is equivalent to $\text{WD}(\rho_x)$ begin generic by 1.2.5.

**Lemma 1.3.2.** Assume $\ell = p$. Let $x$ be a closed point of $\text{Spec} R^\psi_p[1/p]$, resp. of $\text{Spec} R^\psi_p[1/p]$, with residue field $k$. If $\text{Hom}_{k[G_K]}(V_{\rho_x}, V_{\rho_x}(1)) = 0$ then the localization and completion of $R^\psi_p$, resp. of $R^\psi_p$, at $x$ is formally smooth of dimension $d^2([K : Q_p] + 1)$, resp. dimension $(d^2 - 1)([K : Q_p] + 1)$.

**Proof.** As with the proof of 1.3.1, since the maximal pro-$p$ abelian quotient of $G_K$ has $\mathbb{Z}_p$-rank $[K : Q_p] + 1$, the statement for $R^\psi_p$ implies that for $R^\psi_p$ using 1.11. By Tate local duality, $H^0(G_K, \text{ad}(\rho_x)) = \text{Hom}_{k[G_K]}(V_{\rho_x}, V_{\rho_x}(1)) = 0$ is equivalent to $H^2(G_K, \text{ad}(\rho_x)) = 0$. Then 1.15 and part 3. of 1.19 imply $(R^\psi_p)_x^\omega$ is formally smooth over $E$ of dimension $\dim_k Z^1(G_K, \text{ad}(\rho_x))$. Using the local Euler characteristic together with the fact that $H^2(G_K, \text{ad}(\rho_x)) = 0$, we have

$$\dim_k Z^1(G_K, \text{ad}(\rho_x)) = \dim_k H^1(E, \text{ad}(\rho_x)) + d^2 - \dim_H H^0(G_K, \text{ad}(\rho_x)) = d^2([K : Q_p] + 1).$$

□

In fact, it’s easy to see in either of the two above lemmas that a point is smooth on the fixed determinant lifting ring if and only if it is smooth on the nonfixed determinant lifting ring without appealing to 1.11, since $H^0(G_v, \text{ad}(\rho_x)(1)) = H^0(G_v, \text{ad}^0(\rho_x)(1))$.

1.3.3. Assume $\ell = p$. An $n$-dimensional Galois type over $E$ is a representation $\tau : I_K \to \text{Aut}_E(V)$ of $I_K$ on an $n$-dimensional $E$-vector space $V$ with open kernel that extends to a representation of $W_K$. An $n$-dimensional $p$-adic Hodge type over $E$ is a pair $(\psi, \psi')$ of $\text{Fil}^i_{E}$-modules of rank $n$, and $\text{Fil}^i_{E}$ is a decreasing, separated, exhaustive filtration on $\text{D}/K$. Let $\text{ad}(\psi) = \text{End}(\text{D}/K)_{\psi}$. Recall that we set $\text{ad}(\psi) = \text{End}(\text{D}/K)_{\psi}$ and $\text{ad}(\psi^+) = \{ f \in \text{ad}(\psi) \mid f(\text{Fil}_n) \subseteq \text{Fil}_n \text{ for all } i \in \mathbb{Z} \}$. We will say that a $p$-adic Hodge type is $\textbf{regular}$ if $\text{ad}(\psi) = \text{End}(\text{D}/K)_{\psi}$.

Let $\tau : I_K \to \text{Aut}_E(V)$ and $\psi = \text{End}(\text{D}/K)_{\psi}$ be an $n$-dimensional Galois type and $p$-adic Hodge type, respectively, over $E$. Let $A$ be a finite $E$-algebra and $V_A$ be a free $A$-module of rank $n$ with a continuous $A$-linear $G_K$-action such that $V_A$ is a potentially semistable representation. Let $(r_A, N_A)$ be the Weil-Deligne representation attached to $V_A$ (viewed as a representation of $G_K$ on an $n(\dim_E A)$-dimensional $E$-vector space). We say that $V_A$ has $\text{Galois type } \tau$ if $r_{A\psi} = r_{A\psi} \otimes E$. Let $B_{\text{dr}}$ be Fontaine’s ring of deRham periods, and let $D_{\text{dr}}(V_A) = (B_{\text{dr}} \otimes_{E} V_A)^{E/K}$ together with its natural filtration induced from the filtration on $B_{\text{dr}}$. We say that $V_A$ has $p$-adic Hodge type $\psi$ if for each $i \in \mathbb{Z}$, there is an isomorphism of $K \otimes_{Q_p} A$-modules

$$\text{gr}^i D_{\text{dr}}(V_A) \cong \text{gr}^i (D) \otimes_E A.$$

We can now state the following fundamental result of Kisin, [Kis08, Theorem 3.3.4].

**Theorem 1.3.4.** Fix an $n$-dimensional Galois type $\tau$, and an $n$-dimensional $p$-adic Hodge type $\psi = (D, \text{Fil}_E^i)$ over $E$. There is an $\mathcal{O}$-flat quotient $R^\psi_p(\tau, \psi)$ of $R^\psi_p$ such that if $A$ is a finite $E$-algebra, an $E$-algebra morphism $x : R^\psi_p(1/p) \to A$ factors through $R^\psi_p(\tau, \psi)(1/p)$ if and only if $x$ is potentially semistable with Galois type $\tau$ and $p$-adic Hodge type $\psi$.

Moreover, if nonzero, then $\text{Spec} R^\psi_p(\tau, \psi)(1/p)$ is equidimensional of dimension $d^2 + \dim_E \text{ad}(D)/\text{ad}(D)^+$, and admits a open dense formally smooth subscheme.

**Corollary 1.3.5.** Fix an $n$-dimensional Galois type $\tau$, and an $n$-dimensional $p$-adic Hodge type $\psi = (D, \text{Fil}_E^i)$ over $E$. There is an $\mathcal{O}$-flat quotient $R^\psi_p(\tau, \psi)$ of $R^\psi_p$ satisfying the following:

1. If $A$ is any finite $E$-algebra, an $E$-algebra morphism $x : R^\psi_p(1/p) \to A$ factors through $R^\psi_p(\tau, \psi)(1/p)$ if and only if $x$ is potentially semistable with Galois type $\tau$ and $p$-adic Hodge type $\psi$.

2. If nonzero, then Spec $R^\psi_p(\tau, \psi)(1/p)$ is equidimensional of dimension $d^2 - 1 + \dim_E \text{ad}(D)/\text{ad}(D)^+$, and admits a open dense formally smooth subscheme.
3. Let $x$ be a closed point of $\text{Spec} R^{\square, \psi}_\tau(\tau, \psi)[1/p]$, and denote again by $x$ the induced closed point of $\text{Spec} R^{\square}_\tau(\tau, \psi)[1/p]$. Then $x$ is a formally smooth point of $\text{Spec} R^{\square, \psi}_\tau(\tau, \psi)[1/p]$ if and only if it is a formally smooth point of $\text{Spec} R^{\square}_\tau(\tau, \psi)[1/p]$.

Proof. The fact that an $\mathcal{O}$-flat quotient exists satisfying 1. follows from applying [Kis08, Theorem 2.7.6] to $R^{\square, \psi}_\tau[1/p]$. However, we will construct it instead as a quotient of $R^{\square}_\tau(\tau, \psi)$ by fixing the determinant on this ring. Since there is a unique $\mathcal{O}$-flat quotient satisfying 1., this yields the same ring.

There is a unique character $\psi|_{I_K}: I_K \to \mathcal{O}^*$ such that any $p$-adic representation $V$ of Galois type $\tau$ and $p$-adic Hodge type $\psi$ satisfies $\det(V)|_{I_K} = \psi|_{I_K}$. So in order for $R^{\square, \psi}_\tau(\tau, \psi)$ to be nonzero, we must have $\psi|_{I_K} = \psi|_{I_K}$, and we assume this from now on.

Let $K^{ur}$ be the maximal unramified extension of $K$ (inside $\overline{K}$) and let $\Gamma = \text{Gal}(K^{ur}/K)$. Then $\mathcal{G}(\Gamma) = \text{Spf} \mathcal{O}[\Gamma]$ is the CNL$_G$-group that sends a CNL$_G$-algebra $A$ to the set of continuous characters $\chi: \Gamma \to 1 + m_A$. Note that $\mathcal{O}[\Gamma]$ is formally smooth of relative dimension 1 over $\mathcal{O}$.

Let $\rho_{\tau, \psi}$ be the universal $R^{\square}_\tau(\tau, \psi)$-valued lift. If $A$ is a finite $E$-algebra and $\rho_A: G_K \to \text{GL}_d(A)$ is potentially semistable of Galois type $\tau$ and inertial type $\psi$, then $\det(\rho_A)|_{I_K} = \psi|_{I_K}$. The same is then true of $\rho_{\tau, \psi}$. Thus, $\psi^{-1}\det(\rho_{\tau, \psi})$ is a character of $\Gamma$, and we have a CNL$_G$-morphism $\mathcal{O}[\Gamma] \to R^{\square}_\tau(\tau, \psi)$. This action is free by 1.1.12. Then 1.1.11 implies that there is a CNL$_G$-subring $R^{inv}_\tau$ of $R^{\square}_\tau(\tau, \psi)$ such that $R^{inv} \to R^{\square}_\tau(\tau, \psi)$ is formally smooth of relative dimension 1, and such that $R^{inv} \to R^{\square}_\tau(\tau, \psi)$ makes $\text{Spf} R^{\square}_\tau(\tau, \psi)$ a torsor on $\text{Spf} R^{inv}$ for the finite $R^{inv}$-group $(\mu_{p^r})_{R^{inv}}$, were $p^r$ is the largest power of $p$ dividing $d$.

We now wish to show the analogue of part 2. of 1.3.1 for the rings $R^{\square, \psi}_\tau(\tau, \psi)$. For global applications, we will actually only need the fact that WD$(\rho_x)$ generic implies $R^{\square, \psi}_\tau(\tau, \psi)$ is formally smooth, but for completeness we include the converse. Our proof will rely on the following standard lemma, which we will also need for other reasons later.

Lemma 1.3.6. Let $x$ be a closed point of $\text{Spec} R^{\square}_\tau(\tau, \psi)[1/p]$. The tangent space of $\text{Spec} R^{\square}_\tau(\tau, \psi)[1/p]$ at $x$ is isomorphic to

$$Z^1_G(G_K, \text{ad}(\rho_x)) = \ker \left( Z^1(G_K, \text{ad}(\rho_x)) \to H^1(G_K, B_{dR} \otimes_{Q_p} \text{ad}(\rho_x)) \right).$$

If $\det(\rho_x) = \psi$, then this isomorphism identifies the tangent space of $\text{Spec} R^{\square, \psi}_\tau(\tau, \psi)[1/p]$ at $x$ with the subspace

$$Z^1_G(G_K, \text{ad}^0(\rho_x)) = \ker \left( Z^1(G_K, \text{ad}^0(\rho_x)) \to H^1(G_K, B_{dR} \otimes_{Q_p} \text{ad}(\rho_x)) \right).$$

Proof. Let $k$ denote the residue field of $x$. By 1.1.5 and part 1. of 1.1.9, the tangent space of $(R^{\square}_\tau)^{\vee}$ is canonically isomorphic to $Z^1(G_K, \text{ad}(\rho_x))$. Let $c \in Z^1(G_K, \text{ad}(\rho_x))$, and let $\rho_c: G_K \to \text{GL}_d(k[\varepsilon])$ denote the corresponding lift. The cocycle $c$ dies in $H^1(G_K, B_{dR} \otimes_{Q_p} \text{ad}(\rho_x))$ if and only if there is a $G_K$-equivariant isomorphism

$$\rho_c \otimes_{Q_p} B_{dR} \cong (\rho_x \otimes_k k[\varepsilon]) \otimes_{Q_p} B_{dR} \cong (\rho_x \otimes_{Q_p} B_{dR}) \otimes_k k[\varepsilon],$$

and this happens if and only if $\rho_c$ is deRham with $p$-adic Hodge type $\psi$. By Berger’s theorem, this is equivalent to $\rho_c$ being potentially semistable of $p$-adic Hodge type $\psi$. Choosing an extension $L/K$ for which $\rho_c$ is semistable and using the exactness of $D_{st,L}$ (see [Fon94a, Théorème 5.1]), we see that the Galois type of $\rho_c$ is an extension of $\psi$ by itself. Since $\tau$ is a representation of a finite group in characteristic 0, it necessarily splits and $\rho_c$ has Galois type $\tau$. Hence, $c$ lies in the kernel of $Z^1(G_K, \text{ad}(\rho_x)) \to H^1(G_K, B_{dR} \otimes_{Q_p} \text{ad}(\rho))$ if and only if the lift $\rho_c$ is potentially semistable of Galois type $\tau$ and $p$-adic Hodge type $\psi$. By 1.3.4, this is the tangent space of $(R^{\square}_\tau)^{\vee}$.

The proof for the fixed determinant rings is similar, using the fact that $\text{ad}(\rho_x) = \text{ad}^0(\rho_x) \otimes k$ as $[G_K]$-modules.

Proposition 1.3.7. A closed point $x$ of $\text{Spec} R^{\square}_\tau(\tau, \psi)[1/p]$, resp. of $\text{Spec} R^{\square, \psi}_\tau(\tau, \psi)[1/p]$ is formally smooth if and only if WD$(\rho_x)$ is generic. \qed
Proof. The claim for the fixed determinant ring follows from the claim for the nonfixed determinant ring by part 3. of 1.3.5.

Let \( k \) denote the residue field of \( x \). By 1.3.4, we know that \( R_{Q_p}^\square(\tau, v)^{\wedge}_x \) had dimension \( d^2 + \dim_k \text{ad}(D)/\text{ad}(D)^+ \). Since \( \rho_x \) has \( p \)-adic Hodge type \( v \), this is equal to \( d^2 + \dim_k \text{ad}(\text{dR}(\rho_x))/\text{ad}(\text{dR}(\rho_x))^+ \). Since \( \rho_x \) is deRham, we have isomorphisms of filtered \( K \otimes_{Q_p} k \)-modules

\[
\begin{align*}
\text{dR}(\text{ad}(\rho_x)) &= (\text{dR} \otimes_{Q_p} \text{Hom}_k(V_{\rho_x}, V_{\rho_x}))^{G_K} \\
&= (\text{Hom}_{\text{dR} \otimes_{Q_p} k}(\text{dR} \otimes_{Q_p} V_{\rho_x}, \text{dR} \otimes_{Q_p} V_{\rho_x}))^{G_K} \\
&= (\text{Hom}_{\text{dR} \otimes_{Q_p} k}(\text{dR}(\rho_x), \text{dR}(\rho_x)))^{G_K} \\
&= (\text{dR} \otimes_{K} \text{Hom}_{K \otimes_{Q_p} k}(\text{dR}(\rho_x), \text{dR}(\rho_x)))^{G_K} \\
&= (\text{dR} \otimes_{K} \text{ad}(\text{dR}(\rho_x)))^{G_K} \\
&= \text{ad}(\text{dR}(\rho_x)).
\end{align*}
\]

So the dimension of \( R_{Q_p}^\square(\tau, v)^{\wedge}_x \) is \( d^2 + \dim_k \text{dR}(\text{ad}(\rho_x))/\text{dR}(\text{ad}(\rho_x))^+ \).

We now analyse the dimension of the tangent space of \( R_{Q_p}^\square(\tau, v)^{\wedge}_x \), but first introduce some notation. If \( W \) is a finite dimensional \( Q_p \)-vector space with a continuous \( Q_p \)-linear \( G_K \)-action, define the \( Q_p \)-vector spaces as in \([BK, \S3]\):

\[
\begin{align*}
H^1_{dR}(G_K, W) &:= \ker(H^1(G_K, W) \to H^1(G_K, B_{cr}^{\otimes 1} \otimes_{Q_p} W)), \\
H^1_{dR}(G_K, W) &:= \ker(H^1(G_K, W) \to H^1(G_K, B_{cr} \otimes_{Q_p} W)), \\
H^1_{dR}(G_K, W) &:= \ker(H^1(G_K, W) \to H^1(G_K, B_{dR} \otimes_{Q_p} W)).
\end{align*}
\]

By 1.3.6, the tangent space of \( R_{Q_p}^\square(\tau, v)^{\wedge}_x \) has dimension

\[
\dim_k Z^1_H(G_K, \text{ad}(\rho_x)) = d^2 + \dim_k H^1_{dR}(G_K, \text{ad}(\rho_x)) - \dim_k H^0(G_K, \text{ad}(\rho_x)).
\]

So, \( R_{Q_p}^\square(\tau, v)^{\wedge}_x \) is smooth if and only if

\[
\dim_k \text{dR}(\text{ad}(\rho_x))/\text{dR}(\text{ad}(\rho_x))^+ = \dim_k H^1_{dR}(G_K, \text{ad}(\rho_x)) - \dim_k H^0(G_K, \text{ad}(\rho_x)),
\]
equivalently,

\[
(5) \quad \dim_{Q_p} \text{dR}(\text{ad}(\rho_x))/\text{dR}(\text{ad}(\rho_x))^+ = \dim_{Q_p} H^1_{dR}(G_K, \text{ad}(\rho_x)) - \dim_{Q_p} H^0(G_K, \text{ad}(\rho_x)).
\]

The pairing \( (X, Y) \mapsto \text{tr}_{k/Q_p}(\text{tr}(XY)) \) is perfect on \( \text{ad}(\rho_x) \), so induces an isomorphism \( \text{ad}(\rho_x)(1) \cong \text{Hom}_{Q_p}(\text{ad}(\rho_x), Q_p(1)) \). Then, by \([BK, \text{Proposition } 3.8]\),

\[
\begin{align*}
\dim_{Q_p} H^1_{dR}(G_K, \text{ad}(\rho_x)) - \dim_{Q_p} H^0(G_K, \text{ad}(\rho_x)) \\
= \dim_{Q_p} H^1_{dR}(G_K, \text{ad}(\rho_x)) - \dim_{Q_p} H^1_{dR}(G_K, \text{ad}(\rho_x)(1)) - \dim_{Q_p} H^0(G_K, \text{ad}(\rho_x)) \\
= \dim_{Q_p} H^1_{dR}(G_K, \text{ad}(\rho_x)) + \dim_{Q_p} H^1_{dR}(G_K, \text{ad}(\rho_x)(1)) - \dim_{Q_p} H^0(G_K, \text{ad}(\rho_x)(1)) - \dim_{Q_p} H^0(G_K, \text{ad}(\rho_x)).
\end{align*}
\]

Using \([BK, \text{Corollary } 3.8.4]\), this last expression equals

\[
\dim_{Q_p} \text{dR}(\text{ad}(\rho_x))/\text{dR}(\text{ad}(\rho_x))^+ + \dim_{Q_p} D_{ct}(\text{ad}(\rho_x)(1))^{\epsilon=1}.
\]

Plugging this into (5), we see that \( R_{Q_p}^\square(\tau, v)^{\wedge}_x \) is formally smooth if and only if \( D_{ct}(\text{ad}(\rho_x)(1))^{\epsilon=1} = 0 \). This happens if and only if \( WD(\rho_x) \) is generic by part 2. of 1.2.7.

We remark that more thorough investigations of the singular locus in the case \( d = 2 \) and the case \( d = 3 \) and \( K_0 = \mathbb{Q}_p \) are carried out in \([Kis09b, (A.1)]\) and \([Bel14, \S7]\).

1.4. Global Galois deformation rings. Let \( F \) be a number field and let \( \overline{F} \) be a fixed algebraic closure and \( G_F = \text{Gal}(\overline{F}/F) \). For any finite set \( S \) of places of \( F \), we let \( F_S \) denote the maximal extension of \( F \) in \( \overline{F} \) unramified outside of \( S \), and let \( G_{F,S} = \text{Gal}(F_S/F) \). Throughout this subsection, given a continuous representation \( \rho : G_F \to \text{GL}_d(\mathbb{F}) \), if \( v \) is a place of \( F \), we will write \( D_{ct}^\square \) and \( R_v^\square \) for the lifting functor and universal lifting ring, respectively, on \( \text{CNLO}_\rho \), for the local representation \( \rho|_{G_v} : G_v \to \text{GL}_d(\mathbb{F}) \). If \( T \) is any finite set of places of \( F \), then we set \( R_T^\square = \otimes_{v \in T} R_v^\square \), with the completed tensor product taken over \( \mathcal{O} \).
Definition 1.4.1. A global deformation datum is a tuple
\[ S = (S, T, \overline{\rho}, \psi, (\overline{R}_v)_{v \in T}), \]
where
- \( S \) is a finite set of places of \( F \);
- \( T \subseteq S \);
- \( \overline{\rho} : G_{F, S} \to \text{GL}_d(\mathbb{F}) \) is a continuous representation;
- \( \psi : G_{F, S} \to \mathbb{O}^\times \) is a continuous character with \( \psi = \det(\overline{\rho}) \);
- \( \overline{R}_v \) is a deformation quotient of \( R_v^\square \) (see 1.1.6).

Definition 1.4.2. For a global deformation datum \( S = (S, T, \overline{\rho}, \psi, (\overline{R}_v)_{v \in T}) \), a deformation of type \( S \) to a \( \text{CNL}_\mathcal{O} \)-algebra \( A \) is a deformation \([\rho] \) of \( \overline{\rho} \) to \( A \) such that \( \det(\rho) = \psi \) and such that for all \( v \in T \), the map \( R_v^\square \to A \) induced by \( \rho|_{G_v} \) factors through \( \overline{R}_v \). A framed deformation of type \( S \) is a T-framed deformation \([\rho, (\alpha_v)_{v \in T}] \) such that \([\rho] \) is a deformation of type \( S \).

We let \( D_S \) be the set valued functor on \( \text{CNL}_\mathcal{O} \) that takes a \( \text{CNL}_\mathcal{O} \)-algebra \( A \) to the set of deformations of type \( S \). We let \( D_S^\square \) be the set valued functor on \( \text{CNL}_\mathcal{O} \) that takes a \( \text{CNL}_\mathcal{O} \)-algebra \( A \) to the set of T-framed deformations of type \( S \).

If \( D_S \) is representable, we call the representing object the universal type \( S \) deformation ring and denote it by \( R_S^\square \). If \( D_S^\square \) is representable, we call the representing object the universal type \( S \) framed deformation ring and denote it by \( R_S^\square \).

If \( S = (S, T, \overline{\rho}, \psi, (\overline{R}_v)_{v \in T}) \) is a global deformation datum, we set \( R_S^{\text{loc}} = \hat{\bigotimes}_{v \in T} \overline{R}_v \). Note that \( R_S^{\text{loc}} \) is naturally a quotient of \( R_T^\square \).

Proposition 1.4.3. Let \( S = (S, T, \overline{\rho}, \psi, (\overline{R}_v)_{v \in T}) \) be a global deformation datum with \( T \neq \emptyset \).

1. The functor \( D_S^\square \) is representable and there is a canonical \( \text{CNL}_\mathcal{O} \)-morphism \( R_S^{\text{loc}} \to R_S^\square \).
2. Assume \( \text{End}_{F}[G_{F}] (V_{\overline{\rho}}) = \mathbb{F} \). Then \( D_S \) is representable and there is a canonical \( \text{CNL}_\mathcal{O} \)-morphism \( R_S \to R_S^\square \), which is formally smooth of relative dimension \( d^2|T| - 1 \).

Proof. If \( T \neq \emptyset \), then \( D_S^\square \) is representable. For each \( v \in T \), there is a canonical map \( D_S^{\square, T} \to D_S^\square \) given on points by \([\rho, (\alpha_v)_{v \in T}] \mapsto \alpha_v^{-1}(\rho|_{G_v}) \alpha_v \). So \( R_S^{\square, T} \) is canonically an algebra over \( R_T^\square \) and it is easy to see that \( R_S^{\square, T} \to R_S^\square \) represents \( D_S^\square \).

If \( \text{End}_{F}[G_{F}] (V_{\overline{\rho}}) = \mathbb{F} \), then \( D_S^\square \) is representable, and \( D_S \) is represented by the quotient of \( R_S^{\rho} \) by the kernel of the natural map \( R_S^\rho \to R_S^{\square, T} \to R_S^\square \). Moreover, \( D_S^\square \) is a deformation problem (in the sense of 1.1.6), so the map \( R_S \to R_S^\square \) is formally smooth of relative dimension \( d^2|T| - 1 \) by 1.1.7.

Lemma 1.4.4. Let \( S \) be a finite set of places of \( F \) containing all places above \( p \) and above \( \infty \), and let \( T \subseteq S \) contain all places above \( p \), but none above \( \infty \). Let \( \rho : G_{F, S} \to \text{GL}_d(\mathcal{O}) \) be a continuous representation. Set \( \rho_E = \rho \otimes \mathcal{O} E \) and \( \overline{\rho} = \rho_0 \otimes \mathbb{F} \). Assume the following.

(a) \( \text{End}_{F}[G_{F}](V_{\overline{\rho}} \otimes \mathbb{F}) = \mathbb{F} \),
(b) \( \rho|_{G_v} \) is potentially semistable for all \( v \nmid p \).

Let \( \psi = \det(\rho) \). For each \( v \mid p \), we let \( \tau_v \) and \( \nu_v \) be the Galois type and \( p \)-adic Hodge type, respectively, of \( \rho|_{G_v} \) (see 1.2.6).

Let \( S \) be the global deformation datum
\[ S = (S, T, \overline{\rho}, \psi, (\overline{R}_v)_{v \in T}) \]
where
- \( \overline{R}_v = R_v^{\square, \psi}(\tau_v, \nu_v) \) if \( v \mid p \),
- \( \overline{R}_v = R_v^{\square, \psi} \) modulo its \( p \)-torsion if \( v \nmid p \).

Let \( R_S \) be the universal type \( S \) deformation ring on \( \text{CNL}_\mathcal{O} \) and let \( x : R_S[1/p] \to E \) be the \( E \)-algebra morphism induced by \( \rho \).
Then the tangent space of \((R_S)_{x}^{\wedge}\) is isomorphic to
\[
H^{1}_g(G,F_S, \text{ad}^0(\rho_E)) := \ker\left( H^1(G,F_S, \text{ad}^0(\rho_E)) \to \prod_{v|p} H^1(G_v, B_{\text{ad}} \otimes \Q_{p}, \text{ad}^0(\rho_E)) \right).
\]

**Proof.** Denote again by \(x\) the \(E\)-algebra morphisms
- \(R_S^{[1/p]} \to E\) determined by the \(T\)-framed deformation \([\rho, (1)]_{v \in T}\),
- \(R_S^{[1/p]} \to E\) determined by the tuple of local lifts \((\rho_{G_v})_{v \in T}\),
- \(R_S^{\text{loc}}[1/p] \to E\) determined by the tuple of local lifts \((\rho_{G_v})_{v \in T}\),
- \(R_S^{\text{loc}}[1/p] \to E\) determined by the local lift \(\rho_{G_v}\), for each \(v \in T\).

Then \((R_S)_{x}^{\wedge}\) is the quotient of \((R_S^{[1/p]})_{x}^{\wedge}\) by the kernel of the map
\[
(R_S^{[1/p]})_{x}^{\wedge} \to (R_S^{[1/p]})_{x}^{\wedge} \cong (R_S^{\text{loc}}[1/p] \otimes (R_S^{\text{loc}})_{x}^{\wedge}) \to (R_S^{\text{loc}}[1/p] \otimes (R_S^{\text{loc}})_{x}^{\wedge}).
\]
By 1.1.5 (see 1.1.8) and part 2. of 1.1.9, the tangent space of \((R_S^{[1/p]})_{x}^{\wedge}\) is isomorphic to the image of
\[
H^1(G,F_S, \text{ad}^0(\rho_E)) \text{ in } H^1(G,F_S, \text{ad}^0(\rho_E)),
\]
which is \(H^1(G,F_S, \text{ad}^0(\rho_E)) \cong \text{ad}^0(\rho_E) \oplus E[G,F_S]-\text{modules}.
\]
Using (5), we deduce that the tangent space of \((R_S)_{x}^{\wedge}\) is isomorphic to the subspace of \(H^1(G,F_S, \text{ad}^0(\rho_E))\) of cohomology classes \(\gamma\) such that for any cocycle \(c\) representing \(\gamma\) and \(v \in T\), the restriction of the lift \((1 + \varepsilon c)\rho_E\) to \(G_v\) factors through \(R_n\). When \(v \nmid p\), this is no condition. When \(v|p\), this is the condition that the cohomology class \(\gamma\) has trivial restriction to \(H^1(G_v, B_{\text{ad}} \otimes \Q_{p}, \text{ad}^0(\rho_E))\), by 1.3.6.

1.4.5. It will be important for us to understand the minimal number of generators for certain prime ideals in our deformation rings. Before stating the lemma that will allow us to do so, we set up some notation.

Let \(S = (S, T, \bar{\rho}, \bar{\psi}, (R_{v})_{v \in T})\) be a global deformation datum, and assume \(T \neq 0\). Let \(A\) be a \(\text{CNL}_G\)-algebra domain. Let \([\rho, (\alpha_v)] \in D^\emptyset_S(A)\), and let \(x : R_S^{[1/p]} \to A\) denote the induced morphism. Let \(\mathfrak{p} = \ker(x)\). Let \(p^{\text{loc}}\) denote the pullback of \(p\) to \(R_S^{\text{loc}}\). For any \(n \geq 1\), let \(ad_n = \text{Hom}_A(V_{\rho}/m_A^n, V_{\rho}/m_A^n)\) with adjoint \(G,F_S\)-action, and let \(ad_n^{0}\) denote the submodule of elements with trace zero. Denote by \(H^1(G,F_S, \text{ad}_n^{0})\) the image of the map
\[
H^1(G,F_S, \text{ad}_n^{0}) \to H^1(G,F_S, \text{ad}_n)
\]
on cohomology coming from the inclusion \(\text{ad}_n^{0} \to \text{ad}_n\). Finally, let
\[
H^1(S, \text{ad}_n^{0}) = \ker\left( H^1(G,F_S, \text{ad}_n^{0}) \to \prod_{v \in T} H^1(G_v, \text{ad}_n) \right).
\]

**Lemma 1.4.6.** Let \(t_n = \text{Hom}_A(p/p^2,p^{\text{loc}},m_A^n,A/m_A^n)\). Assume that the map \(R_S^{\text{loc}} \to R_S^{1/p} \cong A\) is surjective. Then \(t_n\) fits into an exact sequence
\[
0 \to H^0(G,F_S, \text{ad}_n) \to \prod_{v \in T} H^0(G_v, \text{ad}_n) \to t_n \to H^0(S, \text{ad}_n) \to 0.
\]

**Proof.** Let \(A_n = A/m_A^n\), and let \(A \oplus \varepsilon A_n\) be the local ring given by \(\varepsilon^2 = 0\). Note that \(A \oplus \varepsilon A_n\) is a \(\text{CNL}_G\)-algebra and has maximal ideal \((m_A, \varepsilon)\). View \(A\) as an \(R_S^{\text{loc}}\)-algebra via the map \(R_S^{\text{loc}} \to R_S^{[1/p]} \to A\), and define the set
\[
\mathfrak{X}_n = \{ \phi \in \text{Hom}_{R_S^{\text{loc}}}(R_S^{[1/p]}, A \oplus \varepsilon A_n) \mid \phi \text{ mod } \varepsilon = x \}.
\]
For any \(\phi \in \mathfrak{X}_n\), the restriction of \(\phi\) to \(p\) is of the form \(\varepsilon \lambda\) with \(\lambda \in t_n\). Conversely, given any \(\lambda \in t_n\), we can define \(\phi \in \mathfrak{X}_n\) as follows. Since \(R_S^{\text{loc}} \to R_S^{[1/p]} \cong A\) is surjective, the map \(A \cong R_S^{\text{loc}}/p^{\text{loc}} \to R_S^{1/p}/p^2, p^{\text{loc}}\) is a section of the surjection \(R_S^{[1/p]}/p^2, p^{\text{loc}} \to R_S^{1/p} \cong A\). So we have \(R_S^{1/p}/p^2, p^{\text{loc}} \cong A \oplus \mathfrak{p}/(p^2, p^{\text{loc}})\), and we can set \(\phi = x + \varepsilon \lambda\). It is easy to check that these two constructions are inverse bijections. Thus \(\mathfrak{X}_n\) has a natural \(A\)-module structure and is isomorphic to \(t_n\). We will then show that the \(A\)-module \(\mathfrak{X}_n\) fits into an exact sequence
\[
0 \to H^0(G,F_S, \text{ad}_n) \to \prod_{v \in T} H^0(G_v, \text{ad}_n) \to \mathfrak{X}_n \to H^1(S, \text{ad}_n) \to 0.
\]

By fixing a \(T\)-framed lift \((\rho, (\alpha_v))\) giving rise to our fixed \(T\)-framed deformation \([\rho, (\alpha_v)]\), the set \(\mathfrak{X}_n\) can be identified with an equivalence class of \(T\)-framed lifts \((\rho, (\beta_v))\) to \(A \oplus \varepsilon A_n\) such that
\[
\begin{align*}
(a) & \quad \text{det}(\gamma) = \psi; \\
(b) & \quad \beta_v = \alpha_v + \varepsilon \gamma_v \text{ with } \gamma_v \in M_{d \times d}(A_n) \text{ for each } v \in T;
\end{align*}
\]
(c) the lift $\beta^{-1}_{\nu}(g|_{G_v})$ of $\rho|_{G_v}$ is equal to the extension of scalars of the lift $\alpha^{-1}_{\nu}(\rho|_{G_v})\alpha_v$ from $A$ to $A \oplus \varepsilon A_n$ for each $v \in T$;
(d) two $T$-framed lifts are equivalent if they are conjugate by an element of $1 + \varepsilon M_{d \times d}(A_n)$.

Any $g$ satisfying (a) and (c) is of the form $(1 + \varepsilon)c$ with $c \in \ker(Z^1(G_{F,S}, \text{ad}_n^0) \to \prod_{v \in T} H^1(G_v, \text{ad}_n))$, and the $1 + \varepsilon M_{d \times d}(A_n)$ conjugacy class of $g$ corresponds to the image of $c$ in $H^1_S(G_{F,S}, \text{ad}_n^0)'$. This gives an $A$-module surjection $X_n \to H^1_S(G_{F,S}, \text{ad}_n^0)'$, whose kernel consists of an equivalence class of tuples $(\gamma_v)_{v \in T}$ as in (b), with two tuples $(\gamma_v)_{v \in T}$ and $(\gamma'_v)_{v \in T}$ being equivalent if there is $\delta \in M_{d \times d}(A_n)$ commuting with $\rho \mod m_n^A$ such that $\gamma'_v = \delta \gamma_v$ for all $v \in T$.

1.4.7. We keep the assumptions and notation of 1.4.5. We further assume that $A$ is a discrete valuation ring (so is the ring of integers in a non-Archimedean local field). We let $\lambda$ be a uniformizer for $A$. We also assume that we are given a positive integer $h$, and for each $n \geq 1$, a set $Q_n$ of places of $F$, disjoint from $S$, of cardinality $h$. We let $S_n$ be the global deformation datum

$$S_n = (S \cup Q_n, T, \rho, \psi, (R_v)_{v \in T}).$$

For each $n \geq 1$, let $R_{S_n}$ be the universal framed type $S_n$ deformation ring. Note that $R^\text{loc}_{S_n} = R^\text{loc}_S$. Any type $T$-framed deformation of type $S$ is a $T$-framed deformation of type $S_n$, so there is a natural surjection $R^\text{loc}_{S_n} \to R^\text{loc}_S$ and we let $p_n$ be the pullback of $p$ under this map. Since the diagram

$$
\begin{array}{ccc}
R^\text{loc}_{S_n} & \rightarrow & R^\square_{S_n} \\
\downarrow & & \downarrow \\
R^\text{loc}_S & \rightarrow & R^\square_S
\end{array}
$$

commutes, $p_n$ pulls back to $p^\text{loc}$. In particular, we may consider the $A$-modules $p_n/((p_n)^2, p^\text{loc}, \lambda^n)$ for any $n \geq 1$.

**Lemma 1.4.8.** Assume that the map $R^\text{loc}_S \to R^\square_S/p \cong A$ is surjective, and that if $A$ has characteristic $p$, then $p \nmid d$. Fix a nonnegative integer $h$ and assume that for all $n \geq 1$, there is an $A$-module map

$$(A/\lambda^n)^r \rightarrow H^1_{S_n}(G_{F,S \cup Q_n}, \text{ad}_n^0)$$

with kernel and cokernel of size bounded independently of $n$, where

$$r = h - [F: Q] - \sum_{v \in T} \text{rank}_A H^0(G_v, \text{ad}_n(\rho)).$$

Then for all $n \geq 1$, there is a $A$-module map

$$(A/\lambda^n)^g \rightarrow p_n/((p_n)^2, p^\text{loc}, \lambda^n)$$

with kernel and cokernel of size bounded independently of $n$, with

$$g = h - [F: Q] + |T| - 1.$$

**Proof.** Since $p/((p_n)^2, p^\text{loc}, \lambda^n)$ is a finite cardinality $A$-module annihilated by $\lambda^n$ it is isomorphic to its $A/\lambda^n$-linear dual $t_n = \text{Hom}_A(p/((p_n)^2, p^\text{loc}, \lambda^n), A/\lambda^n)$. By 1.4.6, $t_n$ fits into an exact sequence

$$0 \rightarrow H^0(G_{F,S \cup Q_n}, \text{ad}_n) \rightarrow \prod_{v \in T} H^0(G_v, \text{ad}_n) \rightarrow t_n \rightarrow H^1_{S_n}(G_{F,S \cup Q_n}, \text{ad}_n)' \rightarrow 0.$$

For any $n \geq 1$, the image of the scalar endomorphisms in $\text{ad}_n$ under the natural map $\text{ad}_n \rightarrow \text{ad}_n/\text{ad}_n^0 \cong A/\lambda^n$ is equal to $d(A/\lambda^n)$. Since the characteristic of $A$ does not divide $d$, the index of $d(A/\lambda^n)$ in $A/\lambda^n$ is bounded independently of $n$. From the long exact sequence in cohomology associated to the short exact sequence

$$0 \rightarrow \text{ad}_n^0 \rightarrow \text{ad}_n \rightarrow \text{ad}_n/\text{ad}_n^0 \rightarrow 0,$$

we see that the surjection $H^1_{S_n}(G_{F,S \cup Q_n}, \text{ad}_n^0) \rightarrow H^1_{S_n}(G_{F,S \cup Q_n}, \text{ad}_n)'$ has kernel of size bounded independently of $n$. Thus, the assumptions of the lemma imply we can choose $A$-modules maps

$$(A/\lambda^n)^r \rightarrow H^1_{S_n}(G_{F,S \cup Q_n}, \text{ad}_n)'$$
whose kernels and cokernels are of size bounded independently of \( n \). Since \( t_n \) is annihilated by \( \lambda^n \), we can lift these maps to maps \( \alpha_n : (A/\lambda^n)^r \to t_n \) such that the kernel of \( \alpha_n \) and the cokernel of \( \alpha_n \) have size bounded independently of \( n \).

Applying cohomology to the exact sequence
\[
0 \to \text{ad}(\rho) \xrightarrow{\lambda^n} \text{ad}(\rho) \to \text{ad}_n \to 0,
\]
we see that for each \( v \in T \), we have an injection \( H^0(G_v, \text{ad}(\rho))/\lambda^n \to H^0(G_v, \text{ad}_n) \) with cokernel of size bounded independently of \( n \geq 1 \). Thus, we can define injections \( (A/\lambda^n)^t \to \prod_{v \in T} H^0(G_v, \text{ad}_n) \) with cokernels of size bounded independently of \( n \geq 1 \), where
\[
t = \sum_{v \in T} \text{rank}_A H^0(G_v, \text{ad}(\rho)).
\]
Let \( L \) be the fraction field of \( A \). Since the characteristic of \( L \) does not divide \( d \), we have \( \text{ad}(\rho) \otimes_A L \cong L \oplus (\text{ad}^0(\rho) \otimes_A L) \) for each \( v \in T \), hence
\[
t = |T| + \sum_{v \in T} \text{rank}_A H^0(G_v, \text{ad}^0(\rho)).
\]
Since \( \text{End}_F[G_F](V_\bar{\rho}) = F \), we have \( A/\lambda^n \cong H^0(G_F, \text{ad}_n) \) for all \( n \geq 1 \). And, since the map \( H^0(G_F, \text{ad}_n) \to \prod_{v \in T} H^0(G_v, \text{ad}_n) \) is injective, we can define \( A \)-module maps
\[
\beta_n : (A/\lambda^n)^{t-1} \to \left( \prod_{v \in T} H^0(G_v, \text{ad}_n) \right)/H^0(G_F, \text{ad}_n)
\]
whose kernel and cokernel are bounded independent of \( n \geq 1 \). Then, \( g = r + t - 1 = h - [F : \mathbb{Q}] + |T| - 1 \), and the maps
\[
\alpha_n + \beta_n : (A/\lambda^n)^g \to t_n
\]
have kernel and cokernel of size bounded independently of \( n \geq 1 \).

1.4.9. In this subsection, we specialize to dimension 2. Fix a global deformation datum
\[
S = (S, T, \bar{\rho}, \psi, (\bar{\rho}_v)_{v \in T}),
\]
with \( \bar{\rho} : G_{F,S} \to \text{GL}_2(F) \) satisfying \( \text{End}_F[G_F](V_{\bar{\rho}}) = F \). We assume that \( F \) is large enough to contain the eigenvalues of all elements in the image of \( \bar{\rho} \).

Let \( Q \) be a finite set of places of \( F \), disjoint from \( S \), such that each \( v \in Q \) satisfies
(a) \( \text{Num}(v) \equiv 1 \pmod{p} \),
(b) \( \bar{\rho}(\text{Frob}_v) \) has distinct eigenvalues
For each \( v \in Q \), choose a root \( \alpha_v \) of the characteristic polynomial for \( \bar{\rho}(\text{Frob}_v) \). Let \( S_Q \) be the global deformation datum
\[
S_Q = (S \cup Q, T, \bar{\rho}, \psi, (\bar{\rho}_v)_{v \in T}).
\]
Any type \( S \) deformation is a type \( S_Q \) deformation, so there is a natural \( \text{CNC}_{\mathcal{O}} \) morphism \( R_{S_Q} \to R_S \), which is surjective.

The proof the following lemma is exactly as in [DDT, Lemma 2.44].

**Lemma 1.4.10.** Let \( \rho_{S_Q} : G_{F,S,\mathcal{O}} \to \text{GL}_2(R_{S_Q}) \) be a representative for the universal type \( S_Q \) deformation. Then \( \rho_{S_Q} |_{G_v} \cong \chi \oplus \psi \chi^{-1} \) for a character \( \chi : G_v \to R_{S_Q}^\times \) with \( \chi(\text{Frob}_v) = \alpha_v \).

Note that the characters \( \chi_v \) restricted to inertia have finite \( p \)-power order, so they may be viewed as characters on the maximal \( p \)-power quotient \( \Delta_v \) of \( \mathbb{F}_v^\times \) by local class field theory. Setting \( \Delta = \prod_{v \in Q} \Delta_v \), the characters \( \chi_v \) give a natural \( \mathcal{O}[\Delta] \)-algebra structure to \( R_{S_Q} \). The next lemma follows immediately from 1.4.10.

**Lemma 1.4.11.** The natural surjections \( R_{S_Q} \to R_S \) and \( R_{S_Q}^\square \to R_S^\square \) have kernels \( \alpha R_{S_Q} \) and \( \alpha R_{S_Q}^\square \), respectively, where \( \alpha \) denotes the augmentation ideal of \( \mathcal{O}[\Delta] \).
2. Automorphic theory

We now introduce some notation and assumptions that will be used throughout this section. We denote by $F$ a totally real field, $\mathcal{O}_F$ its ring of integers, and $\mathcal{A}_F$ its ring of adeles. If $S$ is a finite set of places of $F$, we let $\mathcal{A}_{F,S}$ denote $\prod_{v \in S} \mathcal{O}_v$ and $\mathcal{A}_F^\Sigma$ denote $\prod_{v \notin S} \mathcal{O}_v$.

Let $J_F$ denote the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$. Let $E/\mathbb{Q}_p$ be a finite extension inside $\overline{\mathbb{Q}}_p$ containing all embeddings of $F \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathcal{O}$ denote the ring of integers of $E$. We call a pair $\kappa = (\kappa, \psi) \in \mathbb{Z}^{J_F} \times \mathbb{Z}^{J_F}$ an algebraic weight if $k_\tau \geq 2$ for all $\tau$ and $k_\tau + 2w_\tau$ is independent of $\tau \in J_F$. For each finite place $v$ of $F$, we let $m_v$ denote the maximal ideal of $\mathcal{O}_{F_v}$ and $\mathcal{F}_v$ the residue field.

2.1. Quaternionic automorphic forms. Let $D$ denote a quaternion algebra with centre $F$ ramified at all infinite places and split at all places above $p$. Denote by $\Sigma$ the set of finite places at which $D$ ramifies. Fix a maximal order $\mathcal{O}_D$ of $D$ and, for each finite place $v$ at which $D$ is split, an isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v} \cong M_2(\mathcal{O}_{F_v})$ of $\mathcal{O}_{F_v}$-algebras. This determines an isomorphism $D_v \cong \text{GL}_2(F_v)$ sending $(\mathcal{O}_D)_v$ to $\text{GL}_2(\mathcal{O}_{F_v})$. Using this, we identify $(D \otimes_F \mathcal{A}_F^\Sigma)^\times$ with $\text{GL}_2(\mathcal{A}_F^\Sigma)$. Fix a compact open subgroup $\mathcal{U} \subseteq (D \otimes F \mathcal{A}_F^\Sigma)^\times$ such that $\mathcal{U} \subseteq \prod_v (\mathcal{O}_D)_v^\times$.

Let $A$ be a topological $\mathcal{O}$-algebra. For each $\tau \in J_F$, we have an isomorphism $\mathcal{O}_D \otimes_{\mathcal{O}_F, \tau} \mathcal{O} \cong M_2(\mathcal{O})$. Via this isomorphism, given $\tau \in J_F$, $k_\tau \geq 2$ and $w_\tau \in \mathbb{Z}$, we can view

$$\text{Sym}^{k_\tau-2} A^2 \otimes_{\mathcal{O}} \text{det}^{w_\tau} A^2$$

as an $\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v} \cong M_{2 \times 2}(\mathcal{O}_{F_v})$-module, which we denote by $W_{k_\tau, w_\tau}(A)$. Given an algebraic weight $\kappa = (\kappa, \psi) \in \mathbb{Z}^{J_F} \times \mathbb{Z}^{J_F}$, we set $W_{\kappa}(A) = \otimes_{\tau \in J_F} W_{k_\tau, w_\tau}(A)$.

We fix an algebraic weight $\kappa = (\kappa, \psi)$, a character $\theta : F^\times \backslash (\mathcal{A}_F^\Sigma)^\times \rightarrow \mathcal{O}^\times$, and we assume that $z_p$ acts on $W_\kappa(\mathcal{O})$ by $\theta(z)^{-1}$ for all $z \in (\mathcal{A}_F^\Sigma)^\times \cap U$. Let $S_{\kappa, \theta}(U, A)$ denote the $A$-module of functions $f : (D \otimes_F \mathcal{A}_F^\Sigma)^\times \rightarrow W_\kappa(A)$ such that $f(gu) = u^{-1}f(g)$ and $f(zg) = \theta(z)f(g)$ for all $g \in (D \otimes_F \mathcal{A}_F^\Sigma)^\times$, $u \in U$, and $z \in (\mathcal{A}_F^\Sigma)^\times$. If $A = \mathcal{O}$, then we omit it from the notation and simply write $S_{\kappa, \theta}(U)$.

2.1.1. For $g \in (D \otimes_F \mathcal{A}_F^\Sigma)^\times \times (\mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v})^\times$ there is a double coset operator

$$[UgU] : S_{\kappa, \psi}(U, A) \rightarrow S_{\kappa, \psi}(U, A)$$

given by $([UgU]f)(x) = \sum_i g_i f(xg_i)$, where $UgU = \sqcup_i g_i U$. If $A$ is an $E$ vector space, then this double coset operator is defined for any $g \in (D \otimes_F \mathcal{A}_F^\Sigma)^\times$.

Let $S$ be a set of finite places of $F$ containing all places where $D$ is ramified as well as all places where $D$ is split and $U_v \neq \text{GL}_2(\mathcal{O}_{F_v})$. Let $\mathcal{T}_{\text{univ}}^S$ denote the polynomial ring over $\mathcal{O}$ in the variables $\{T_v, S_v \mid v \notin S, v \neq \infty\}$. Note that for any finite set $S'$ of finite places containing $S$, $\mathcal{T}_{\text{univ}}^{S'} \subseteq \mathcal{T}_{\text{univ}}^S$. We define an action of $\mathcal{T}_{\text{univ}}^S$ on $S_{\kappa, \theta}(U, A)$ by letting $T_v$ and $S_v$ act via the double coset operators

$$U \begin{bmatrix} \overline{\omega}_v & 1 \\ 1 & \overline{\omega}_v \end{bmatrix} U$$

and

$$U \begin{bmatrix} \overline{\omega}_v & \omega_v \\ \omega_v & \overline{\omega}_v \end{bmatrix} U,$$

respectively, where $\omega_v$ is any choice of uniformizer for $F_v$ (and this is independent of the choice). We denote the $A$-algebra generated by image of $\mathcal{T}_{\text{univ}}^S$ in $\text{End}_A(S_{\kappa, \theta}(U, A))$ by $\mathcal{T}_{\kappa, \theta}^S(U, A)$. Note that the image of $S_v$ in $\mathcal{T}_{\kappa, \theta}^S(U, A)$ is $\theta(\omega_v)$. We again denote by $T_v$ the image of $T_v$ in $\mathcal{T}_{\kappa, \theta}^S(U, A)$. If $A = \mathcal{O}$, then we omit it from the notation and simply write $\mathcal{T}_{\kappa, \theta}^S(U)$.

We say a maximal ideal $m$ of $\mathcal{T}_{\text{univ}}^S$ is Eisenstein if $T_v - 2 \in m$ for all but finitely many $v$ that split in some fixed abelian extension of $F$. We say a maximal ideal of $\mathcal{T}_{\kappa, \theta}^S(U)$ is Eisenstein if it pulls back to an Eisenstein maximal ideal of $\mathcal{T}_{\text{univ}}^S$.

2.1.2. Let $Q$ be a set of finite places of $F$ such that for every $v \in Q$, $D$ is split and $U_v = \text{GL}_2(\mathcal{O}_{F_v})$. For each $v \in Q$, fix a quotient $\Delta_v$ of $\mathbb{F}_p^\times$ of $p$-power order, and set $\Delta = \prod_{v \in Q} \Delta_v$. We define open subgroups
$U_\Delta \subseteq U_0 \subseteq U$ by $(U_\Delta)_v = (U_0)_v = U_v$ for $v \not\in Q$, and

$$(U_0)_v = \left\{ \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \in \GL_2(\mathcal{O}_{F_v}) : c_v \in \mathfrak{m}_v \right\}$$

$$(U_\Delta)_v = \left\{ \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \in \GL_2(\mathcal{O}_{F_v}) : c_v \in \mathfrak{m}_v \text{ and } a_v d_v^{-1} \mapsto 1 \in \Delta_v \right\}$$

for $v \in Q$. The map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \prod_{v \in Q} a_v d_v^{-1}$ defines an isomorphism $U_0/U_\Delta \cong \Delta$. We get an action of $\Delta$ on $S_{\kappa,\theta}(U_\Delta)$ as follows. For any $\delta \in \Delta$, choose a lift $g_\delta \in U_\Delta$ of $\delta$. The double coset operator $[U_\Delta g_\delta U_\Delta]$ is independent of the lift $g_\delta$, and we denote it by $(\delta)$.

For each $v \in Q$, we fix a choice of uniformizer $\varpi_v$ for $F_v$. We let $T^S_{\text{univ}}$ be the polynomial ring over $T^S_{\text{univ}}$ in the variables $\{U_{\varpi_v} \}_{v \in Q}$. We define an action of $T^S_{\text{univ}}$ on $S_{\kappa,\theta}(U_0)$ and $S_{\kappa,\theta}(U_\Delta)$ by letting $U_{\varpi_v}$ act via the double coset operators

$$\left[U_0 \begin{pmatrix} \varpi_v & 1 \\ & 1 \end{pmatrix} U_0 \right] \quad \text{and} \quad \left[U_\Delta \begin{pmatrix} \varpi_v & 1 \\ & 1 \end{pmatrix} U_\Delta \right],$$

respectively. The natural inclusion $S_{\kappa,\theta}(U_0) \subseteq S_{\kappa,\theta}(U_\Delta)$ is $T^S_{\text{univ}}$-equivariant. We let $T^S_{\kappa,\theta}(U_0)$ and $T^S_{\kappa,\theta}(U_\Delta)$ denote the images of $T^S_{\text{univ}}$ in $S_{\kappa,\theta}(U_0)$ and $S_{\kappa,\theta}(U_\Delta)$, respectively.

Let $\mathfrak{m}$ be a maximal ideal of $T^S_{\kappa,\theta}(U)$, and denote again by $\mathfrak{m}$ its pullback to $T^S_{\text{univ}}$ and $T^S_{\text{univ}}$. Assume that for every $v \in Q$, the polynomial $X^2 - T_v X + S_v \text{Nm}(v) \mod \mathfrak{m}$ has distinct $\mathbb{F}$-rational eigenvalues $\alpha_v$ and $\beta_v$. For each $v \in Q$, let $\tilde{\alpha}_v$ be a choice of lift of $\alpha_v$ to $\mathcal{O}$, and let $\mathfrak{m}_Q$ be the ideal in $T^S_{\text{univ}}$ generated by $\mathfrak{m}$ and the elements $U_{\varpi_v} - \tilde{\alpha}_v$ for each $v \in Q$.

**Lemma 2.1.3.**

1. The ideal $\mathfrak{m}_Q$ induces proper maximal ideals in $T^S_{\kappa,\theta}(U_0)$ and $T^S_{\kappa,\theta}(U_\Delta)$.
2. If $\text{Nm}(v) \not\equiv (\alpha_v \beta_v^{-1})^{\pm 1} \pmod{\varpi}$, then the inclusion $S_{\kappa,\theta}(U) \subseteq S_{\kappa,\theta}(U_0)$ induces an isomorphism $S_{\kappa,\theta}(U)_m \cong S_{\kappa,\theta}(U_\Delta)_m$ of $T^S_{\text{univ}}$-modules.
3. Assume that for any $t \in (D \otimes_F \mathbb{A}_F)^\times$, we have

$$U((A_F^\infty)^X \cap t^{-1} D^X \otimes \mathbb{A})/F^X = 1.$$  

Then $S_{\kappa,\theta}(U_\Delta)_m$ is a finite free $\mathcal{O}[[\Delta]]$-module and the trace map

$$\sum_{\delta \in \Delta} \langle \delta \rangle : S_{\kappa,\theta}(U_\Delta)_m \rightarrow S_{\kappa,\theta}(U_0)_m$$

induces an isomorphism from the $\Delta$-coinvariants of $S_{\kappa,\theta}(U_\Delta)_m$ to $S_{\kappa,\theta}(U_0)_m$.

**Proof.** Parts 1. and 2. are [Kis09b, Lemma 2.1.7], and part 3. is [Kis09b, Lemma 2.1.4] (there is a running assumption that $p > 2$ in [Kis09b], but this is not used in the proofs of the above mentioned lemmas).

### 2.2. Automorphic Galois representations.

We keep all assumptions and notations of the previous subsection. In particular $\kappa$ is an algebraic weight, $D$ is a totally definite quaternion algebra with centre $F$, $U$ is a compact open subgroup of $(D \otimes_F \mathbb{A}_F)^\times$, and $S$ is a finite set of places of $F$ containing all those at which $D$ is ramified, $U_v$ is not maximal compact, as well as all places above $p$.

**Theorem 2.2.1.** Let $\pi$ be a regular algebraic cuspidal representation of $\GL_2(\mathbb{A}_F)$, and fix an isomorphism $\iota : \mathbb{C} \cong \mathbb{C}_p$. Let $\kappa = (k, \mathbf{w}) \in \mathbb{Z}_F^{J_F} \times \mathbb{Z}_F^{J_F}$ be the algebraic weight such that for every $\tau \in J_F$, if $v$ denotes the infinite place corresponding to $\tau^{-1}$, then $\pi_v$ is the discrete series representation with lowest weight $k_{\tau} - 1$ and central character $z \mapsto \text{sgn}(z)^{k_{\tau}} |z|^{2k_{\tau}-2w_{\tau}}$. There is a finite extension $L$ of $\mathbb{Q}_p$ containing all embedding of $F$ into an algebraic closure of $L$ and a continuous absolutely irreducible representation

$$\rho_{\pi,\iota} : G_F \rightarrow \GL_2(L)$$

such that the following hold.

1. For any $v | p$, $\rho_{\pi,\iota}|_{G_v}$ is potentially semistable. Moreover, if $\tau \in J_F$ gives rise to $v$, viewing $L$ as an $F_v \otimes_{\mathbb{Q}_p} L$-algebra via $\tau \otimes 1$, the two-dimensional filtered $L$-vector space $D_{\text{dR}}(V_{\rho_{\pi,\iota}}) \otimes (F_v \otimes L)$ $L$ has filtration jumps at $w_{\tau}$ and $k_{\tau} + w_{\tau} - 1$. 

2. For every place $v$ of $F$,
\[ \text{WD}(\rho_{\pi,a}|_{G_v})^{F-ss} \cong \text{irr}_{F_v}(\pi_v). \]

3. Set $w = k_\tau + 2w_\tau$ (which is independent of $\tau \in J_F$). If $v$ is a finite place at which $\rho_{\pi,a}|_{G_v}$ is unramified, then $\rho_{\pi_a}(\text{Frob}_v)$ is algebraic over $\mathbb{Q}$ and has complex absolute value $\text{Nm}(v)^{\frac{2}{w_\tau - 1}}$ under any complex embedding.

For the existence of $\rho_\pi$ as well as properties 1. and 2., see [Ski09, §1] and the references contained there. That $\rho_\pi$ is absolutely irreducible follows from an argument of Ribet (see [Tay, Proposition 3.1]). Part 3. follows from [Bli, Theorem 1].

For the remainder of this subsection we will assume that $D$ is split at all finite places (hence $[F : \mathbb{Q}]$ is even). Everything we say remains true without this assumption (with suitable modification), but this will suffice for our purposes.

**Corollary 2.2.2.** Let $f \in S_{\kappa, \rho}(U, \overline{\mathbb{Q}}_p)$ be an eigenfunction for $T_{\kappa, \rho}^S(U, \overline{\mathbb{Q}}_p)$ and let $\lambda : T_{\kappa, \rho}^S(U, \overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ denote the corresponding $\overline{\mathbb{Q}}_p$-algebra homomorphism. Assume that if $\kappa = \{(2, \ldots, 2), \mathbf{w}\}$, then $f$ does not factor through the reduced norm. There is a finite extension $L/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ and a continuous absolutely irreducible representation
\[ \rho_f : G_{F,S} \to \text{GL}_2(L) \]
such that the following hold.

1. $\det(\rho_f) = \theta^{-1}$.
2. For any $v \notin S$, $\text{tr}(\rho_f(\text{Frob}_v)) = \chi(T_v)$.
3. Set $w = k_\tau + 2w_\tau$ (which is independent of $\tau \in J_F$). For any $v \notin S$, the eigenvalues of $\rho(\text{Frob}_v)$ are algebraic over $\mathbb{Q}$ and have complex absolute value $\text{Nm}(v)^{\frac{2}{w_\tau - 1}}$ under any complex embedding.
4. For any $v | p$, $\rho_f$ is potentially semistable. If $\tau \in J_F$ gives rise to $v$, viewing $E$ as an $F_v \otimes_{\mathbb{Q}_p} E$-algebra via $\tau \otimes 1$, the two-dimensional filtered $E$-vector space $D_{\text{dR}}(V_{\rho_f}) \otimes (F_v \otimes E)$ has filtration jumps at $w_\tau$ and $k_\tau + w_\tau - 1$.
5. For any finite $v$, the Weil–Deligne representation $\text{WD}(\rho_f|_{G_v})$ is generic (in the sense of 1.2.2). If $\pi_v$ has Iwahori fixed vectors, then $I_{F_v}$ acts trivially on $\text{WD}(\rho_f|_{G_v})$.

**Proof.** Fix $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$. Since $f$ is an eigenfunction and doesn’t factor through the reduced norm, it’s image under the Jacquet–Langlands correspondence (see [Tay06, Lemma 1.3]) generates a regular algebraic cuspidal automorphic representation $\pi$ of $\text{GL}_2(\mathbb{A}_F)$ of weight $\kappa$. Part 2. follows from part 2. of 2.2.1 and a simple computation of the action of $T_\kappa$ on an unramified principal series. Parts 3. and 4. are immediate from parts 3. and 1. of 2.2.1. The second sentence in part 5. is immediate from 2. of 2.2.1. It remains to show the first sentence of part 5.. Since any cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ is globally generic, all of the local representations $\pi_v$ are generic. This implies $\text{WD}(\rho_f|_{G_v})^{F-ss}$ is generic for every finite place by 1.2.3, which in turn implies that $\text{WD}(\rho_f|_{G_v})$ is generic. \(\square\)

2.2.3. We fix an eigenfunction $f \in S_{\kappa, \rho}(U, E)$ for $T_{\kappa, \rho}^S(U)$, and let
\[ \lambda_f : T_{\kappa, \rho}^S(U) \to E \]
de note the corresponding $\mathcal{O}$-algebra morphism. The kernel of $\lambda_f$ is contained in a unique maximal ideal $\mathfrak{m}$. Choosing a $\mathcal{O}$-lattice for $\rho_f$ and reducing modulo $\mathfrak{m}$, we obtain a representation
\[ \overline{\rho} : G_{F,S} \to \text{GL}_2(\mathbb{F}). \]
If $\mathfrak{m}$ is non-Eisenstein, $\overline{\rho}$ is irreducible and (up to isomorphism) $\overline{\rho}$ does not depend on the choice of $\mathcal{O}$-lattice in the representation space of $\rho_f$.

Since $S$ contains all places $v$ such that $U_v \neq \text{GL}_2(\mathcal{O}_{F_v})$, there is a decomposition
\[ T_{\kappa, \rho}^S(U)_m \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \cong \bigoplus_{f'} \overline{\mathbb{Q}}_p \]
where the direct sum runs over all eigenfunctions $f' \in S_{\kappa, \rho}(U)_m \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \subseteq S_{\kappa, \rho}(U, \overline{\mathbb{Q}}_p)$. By 2.2.2, we get a representation
\[ \rho : G_{F,S} \to \text{GL}_2(T_{\kappa, \rho}^S(U)_m \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p) \]
with \( \det(\varrho) = \theta e^{-1} \) and \( \mathrm{tr}(\varrho(Frob_v)) = T_v \) for every \( v \notin S \). By Chebotarev density, the pseudorepresentation \( \sigma \mapsto \mathrm{tr}(\varrho(\sigma)) \) takes values in \( \mathrm{GL}_2(\mathbb{T}_{\kappa, \theta}(U)_m) \). Since \( \mathfrak{p} \) is absolutely irreducible, a theorem of Nyssen and Rouquier (see [Nys96, Theorems 1 and 2]) implies there is a continuous representation
\[
\rho_m : G_{F, S} \rightarrow \mathrm{GL}_2(\mathbb{T}_{\kappa, \theta}(U)_m)
\]
satisfying \( \det(\rho_m) = \theta e^{-1} \) and \( \text{tr}(\rho_m(Frob_v)) = T_v \) for every \( v \notin S \). Moreover, if \( L \) is a finite extension of \( E \), and \( \lambda : \mathbb{T}_{\kappa, \theta}(U)[1/p] \rightarrow L \) is an \( E \)-algebra homomorphism, then \( \lambda \) is the eigensystem for some \( f \in S_{\kappa, \theta}(U)_m \otimes \mathcal{O} L \), and induces a representation
\[
\rho_{\lambda} : G_{F, S} \rightarrow \mathrm{GL}_2(\mathbb{T}_{\kappa, \theta}(U)_m) \xrightarrow{\lambda} \mathrm{GL}_2(L)
\]
satisfies the conclusions of 2.2.2.

2.2.4. We now further assume that \( U_v \) contains the Iwahori subgroup for every \( v \nmid p \). Let \( S_f = S \setminus \{ v | \infty \} \) and let \( \psi = \theta e^{-1} \). We consider the global deformation datum (see 1.4.1)
\[
\mathcal{S} = (S, S_f, \mathfrak{p}, \psi, (R_v)_{v \in S_f})
\]
where
- if \( v \nmid p \), then \( \overline{R}_v \) is \( R_v \otimes_{\mathbb{Q}_p} \mathbb{F}_p \) modulo its \( p \)-torsion;
- if \( v | p \), then \( \overline{R}_v = R_v \otimes_{\mathbb{Q}_p} (R_v, \mathcal{V}_v) \) (see 1.3.5) where \( \mathcal{V}_v = D_{\text{cr}}(\rho_f|_{G_v}) \) and \( \tau_v = 1 \) is trivial.

Let \( R_S \) be the universal type \( S \) deformation ring on \( \text{CNL}_{\mathcal{O}} \), and let \( R_{S, \text{loc}} = \bigotimes_{v \in S_f} \overline{R}_v \).

Lemma 2.2.5. 1. The deformation
\[
\rho_m : G_{F, S} \rightarrow \mathrm{GL}_2(\mathbb{T}_{\kappa, \theta}(U)_m)
\]
of \( \mathfrak{p} \) is of type \( S \) and the induced \( \text{CNL}_{\mathcal{O}} \)-morphism \( R_S \rightarrow \mathbb{T}_{\kappa, \theta}(U)_m \) is surjective.

2. Letting \( x : R_S^{\text{int}}[1/p] \rightarrow E \) denote the \( E \)-algebra morphism induced by the local lifts \( \rho_f|_{G_v} \), the localization and completion \( (R_S^{\text{loc}})_{\mathcal{O}_x} \) at \( x \) is formally smooth over \( E \) of dimension \( 3|S_f| + [F : \mathbb{Q}] \).

3. Let \( x : R_S[1/p] \rightarrow E \) denote the \( E \)-algebra morphism induced by the deformation \( \rho_f \) of \( \mathfrak{p} \). The localization and completion \( (R_S)_{\mathcal{O}_x} \) acts freely on \( (S_{\kappa, \theta}(U)_m)_{\mathcal{O}_x} \) if and only if \( (R_S)_{\mathcal{O}_x} \neq \mathbb{E} \).

Proof. Since \( \mathbb{T}_{\kappa, \theta}(U)_m \) is generated over \( \mathcal{O} \) by \( T_v \) with \( v \notin S \), the \( \text{CNL}_{\mathcal{O}} \)-morphism \( R_{\mathfrak{p}} \rightarrow \mathbb{T}_{\kappa, \theta}(U)_m \) induced by the deformation \( \rho_m \) of \( \mathfrak{p} \) is surjective by Chebotarev density. It factors through \( R_{\mathfrak{p}}^{\text{int}} \) by part 1. of 2.2.2. It remains to check that the local lifts \( \rho_m|_{G_v} \) factor through \( \overline{R}_v \) for each \( v \in S_f \). For \( v \nmid p \), this is immediate since \( \mathbb{T}_{\kappa, \theta}(U)_m \) is \( \mathcal{O} \)-flat. For \( v \mid p \), we first note that since \( \mathbb{T}_{\kappa, \theta}(U)_m \) is finite flat over \( \mathcal{O} \) and reduced, it suffices to check that for any \( E \)-algebra morphism \( \lambda : \mathbb{T}_{\kappa, \theta}(U)_m[1/p] \rightarrow L \), with \( L \) a finite extension of \( E \), the representation
\[
\rho_{\lambda} : G_{F, S} \xrightarrow{\rho_m} \mathrm{GL}_2(\mathbb{T}_{\kappa, \theta}(U)_m) \xrightarrow{\lambda} \mathrm{GL}_2(L)
\]
induces an \( E \)-algebra morphism \( R_{\mathfrak{p}}^{\text{int}}[1/p] \rightarrow L \) that factors through \( R_{\mathfrak{p}}^{\text{loc}}(\mathcal{V}_v, \tau_v) \). This happens if and only if \( \rho_{\lambda} \) is potentially semistable with \( p \)-adic Hodge type \( \mathcal{V}_v \) and Galois type \( \tau_v \). Since \( \rho_{\lambda} \) satisfies the conclusions of 2.2.2, it is potentially semistable. Moreover, it has \( p \)-adic Hodge type \( \mathcal{V}_v \), since this property only depends on the associated graded, which is determined up to isomorphism by the weight \( \kappa \) by part 4. of 2.2.2. The fact that it has trivial Galois type follows from the second sentence of part 5. of 2.2.2 since \( U_v \) contains the Iwahori subgroup.

We now show part 2. If \( v \nmid p \), then \( \overline{R}_v \) is \( \mathcal{O} \)-flat of relative dimension 3 by part 1. of 1.3.1. If \( v \mid p \), then \( \overline{R}_v \) is also \( \mathcal{O} \)-flat by 1.3.5. Moreover, letting \( D = D_{\text{cr}}(\rho_f|_{G_v}) \), it has relative dimension \( 3 + \dim_{\mathbb{Q}_p} \text{ad}(D)/\text{ad}(D)^+ \) (in the notation of 1.3.3). Using the fact that for every embedding \( \tau : F_v \rightarrow E \), the two dimensional filtered \( E \)-vector space \( D \otimes_{(F_v \otimes E)} E \) (viewing \( E \) as a \( F_v \otimes \mathbb{Q}_p \)-\( E \)-algebra via \( \tau \otimes 1 \)) has two distinct jumps in its filtration, it is easy to check \( \dim_{\mathbb{Q}_p} \text{ad}(D)/\text{ad}(D)^+ = [F_v : \mathbb{Q}_p] \). Then \( R_{\mathfrak{p}}^{\text{loc}} \) is \( \mathcal{O} \)-flat of relative dimension \( 3|S_f| + [F : \mathbb{Q}] \). To show that \( (R_{\mathfrak{p}}^{\text{loc}})_{\mathcal{O}_x} \) is formally smooth over \( E \), it suffices to show that \( (\overline{R}_v)_{\mathcal{O}_x} \) is formally smooth over \( E \) for each \( v \in S_f \) (see [Kis09a, Lemma 3.4.12]). This follows from part 5. of 2.2.2, part 2. of 1.3.1, and 1.3.7.

For part 3., we note that \( (S_{\kappa, \theta}(U)_m)_{\mathcal{O}_x} \) is a product of fields corresponding to eigenforms in the same \( \mathbb{T}_{\kappa, \theta}(U) \)-eigenclass, so \( (R_S)_{\mathcal{O}_x} \) acts diagonally on this product of fields, which implies it acts through its surjection to its residue field \( E \). \( \square \)
2.2.6. We now let $Q$ be a finite set of places of $F$, disjoint from $S$ such that for each $v \in Q$
- $\text{Nm}(v) \equiv 1 \pmod{p}$,
- $\overline{\rho}(\text{Frob}_v)$ has distinct $\mathbb{F}$-rational eigenvalues.

For each $v \in Q$ we choose an eigenvalue $\alpha_v$ of $\overline{\rho}(\text{Frob}_v)$. For each $v \in Q$, let $\Delta_v$ be the maximal $p$-power quotient of $\mathbb{F}_e^+$, and let $\Delta = \prod_{v \in Q} \Delta_v$. Let $U_\Delta$ and $T_{n,\theta}^{S,Q}(U_\Delta)$ be the open compact subgroup of $U$ and the Hecke algebra, respectively, of 2.1.2. We chose some $\tilde{\alpha}_v \in \mathbb{O}$ that reduces to $\alpha_v \pmod{\varpi}$, and let $m_Q$ be the ideal of $T_{n,\theta}^{S,Q}(U_\Delta)$ generated by $m$ and $U_{\varpi,v} - \tilde{\alpha}_v$ for each $v \in Q$. By part 1. of 2.1.3, $m_Q$ is a maximal ideal of $T_{n,\theta}^{S,Q}(U_\Delta)$.

We continue to let

$$S = (S, S_f, \overline{\rho}, \psi, (\overline{\rho}_v)_{v \in S_f})$$

be the global deformation datum as above 2.2.4, and let $S_Q$ be the global deformation datum

$$S_Q = (S \cup Q, S_f, \overline{\rho}, \psi, (\overline{\rho}_v)_{v \in S_f}).$$

We let $R_{S_Q}$ be the universal type $S_Q$-deformation ring. Recall that the choice of $\alpha_v$ for each $v \in Q$ gives $R_{S_Q}$ the structure of a $\mathcal{O}[\Delta]$-module (see 1.4.9).

Lemma 2.2.7. There is a type $S_Q$ deformation

$$\rho_{m_Q} : G_{F,S} \to \text{GL}_2(T_{n,\theta}^{S,Q}(U_\Delta))$$

of $\overline{\rho}$ such that the induced CNL-$\mathcal{O}$-morphism $R_{S_Q} \to T_{n,\theta}^{S,Q}(U_\Delta)_{m_Q}$ is a surjection of $\mathcal{O}[\Delta]$-algebras.

Proof. Denote again by $m_Q$ the pullback of $m_Q$ to $T_{n,\theta}^{S,Q}(U_\Delta) \subseteq T_{n,\theta}^{S,Q}(U_\Delta)$. The existence of the type $S_Q$-deformation with $m_Q$ as in 1.4.10, this CNL-$\mathcal{O}$-morphism sends $\chi_v(\text{Frob}_v)$ to $U_{\varpi,v}$. This implies that the the elements $\chi_v(\delta)$ have image $\langle \delta \rangle$ in $T_{n,\theta}^{S,Q}(U_\Delta)_{m_Q}$ by our definition of the isomorphism $U_0/\Delta \cong \Delta$, and that the $\mathcal{O}[\Delta]$-algebra structures coincide. □

3. Patching and the main theorems

Throughout this section $E$ will denote a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer $\varpi$, and residue field $\mathbb{F}$. Recall that if $\Lambda$ is a complete local Noetherian ring with residue field $\mathbb{F}$, then we let CNL-$\mathcal{O}$ denotes the category of complete Noetherian local $\Lambda$-algebras $A$ such that the structure map $\Lambda \to A$ induces an isomorphism $\mathbb{F} \cong A/m_A$, and whose morphisms are local $\Lambda$-algebra morphisms.

3.1. Patching.

Proposition 3.1.1. Fix nonnegative integers $h$ and $j$. Let $S = \mathcal{O}[[z_1, \ldots, z_j]]$ and let $S_\infty = S[[\mathbb{Z}_p/h]]$. Let $A$ denote its augmentation ideal of $S_\infty$, and for any $n \geq 1$, we let $a_n$ be the kernel of the natural map $S_\infty \to \mathcal{O}[[\mathbb{Z}_p/p^n/h]]$.

Assume we have the following initial data:
- a CNL-$\mathcal{O}$-algebra $B$;
- a local $B$-algebra $R$ that is also a local $S$-algebra;
- an $R$-module $M$;
- a prime ideal $p$ of $R$.

Let $p_B$ denote the pullback of $p$ to $B$. We assume the initial data is subject to the following conditions:

(a) $p$ contains $z_1, \ldots, z_j$ and $R/p \cong \mathcal{O}$;
(b) $M$ is finite free over $S$;
(c) $B_{pp_B}$ is regular.
(d) $\dim(B)_{pp_B} = j + h$.

For each $n \geq 1$, we assume we are given the following level $n$ data:
- a local $B$-algebra $R_n$, that is also a local $S_\infty$-algebra;
- an $R_n$-module $M_n$;
- a surjection of local rings $R_n \to R$;
– a surjection of $R_n$-modules $M_n \to M$ (with the $R_n$-module structure on $M$ coming via the surjection $R_n \to R$).

We let $\mathfrak{p}_n$ be the pullback of $\mathfrak{p}$ to $R_n$. We assume that the level $n$ data is subject to the following conditions:

(e) the $\mathbb{F}$-dimension of $\mathfrak{m}_{R_n}/\mathfrak{m}_{R_n}^2$ is bounded independently of $n$;
(f) the pullback of $\mathfrak{p}_n$ to $R_\infty$ is $\mathfrak{p}_B$;
(g) the finite cardinality $\mathcal{O}$-module $\mathfrak{p}_n/(\mathfrak{p}_n^2, \mathfrak{p}_B, \varpi^n)$ has size bounded independently of $n$;
(h) $a \in \ker(R_n \to R)$;
(i) $M_n/\mathfrak{a}_n$ is free over $S_\infty/\mathfrak{a}_n$;
(j) the surjection $M_n \to M$ has kernel $\mathfrak{a}_M n$.

Then $M^n_0$ is a (finite) free $R_\infty^n$-module.

Proof. We view $R$ as an $S_\infty$-module by letting $a$ annihilate $R$. By (e) we can, and do, fix a positive integer $k$ such that $\dim_{\mathfrak{m}_{R_n}/\mathfrak{m}_{R_n}^2} \leq k$ for all $n \geq 1$.

For any local ring $A$ and positive integer $r$, we let $\mathfrak{m}_A^{(r)}$ be the ideal in $A$ generated by elements of $\mathfrak{m}_A$ that are $r$th powers. Note that if $A$ is Noetherian, then $\mathfrak{m}_A^{(r)}$ is an open ideal. Let $s$ denote the $S$-rank of $A$. For any $m \geq 1$, we set $\rho_m = smp^k$. We note that $\mathfrak{m}_R^{(m)}$ annihilates $M/\mathfrak{m}_R^n$ for every $m \geq 1$. Indeed, any $a \in \mathfrak{m}_R$ defines a nilpotent endomorphism of $M/m_S^n$, so $a^* M \subseteq m_S M$ and $a^{\infty} M \subseteq m^n_S M$. Hence, $M/m_S^n$ is naturally an $R/(\mathfrak{m}^n_S, \mathfrak{m}_R^{(m)})$-module.

For $m \geq 1$, we define a *patching datum of level $m$* to be

(i) A surjection of local $S_\infty/\mathfrak{b}_m$ algebras $D \to R/(\mathfrak{m}_S^n, \mathfrak{m}_R^{(m)})$ with the local ring $D$ satisfying
   - $\dim_{\mathfrak{m}_D/\mathfrak{m}_D^2} \leq k$,
   - $\mathfrak{m}_D^{(r_m)} = 0$.

(ii) A local $\mathcal{O}$-algebra map $B \to D$.

(iii) A surjection $L \to M/\mathfrak{m}_S^n$ of $D$-modules (where the $D$-module structure on $M/\mathfrak{m}_S^n$ comes via the surjection (i)), such that
   - $L$ that is finite free over $S_\infty/\mathfrak{b}_m$,
   - the kernel of $L \to M/\mathfrak{m}_S^n$ is $aL$.

We denote a patching datum of level $m$ simply by the pair $(D, L)$. A morphism $(D, L) \to (D', L')$ of patching datum of level $m$ is a morphism of $S_\infty/\mathfrak{b}_m$-algebras $D \to D'$ commuting with the $B$-algebra structure, and a morphism $L \to L'$ of $D$-modules (with the $D$-module structure on $L'$ via the map $D \to D'$) commuting with the maps to $M/\mathfrak{m}_S^n$.

Note that if $n \geq m$, then a patching datum $(D, L)$ of level $n$ naturally defines a patching datum of level $m$ by $(D/(\mathfrak{b}_m, \mathfrak{m}_D^{(r_m)}), L/\mathfrak{b}_m)$, and we denote it by $(D, L)$ mod $m$.

Let $n \geq m \geq 1$. We explain how to define a patching datum $(D_{n,m}, L_{n,m})$ of level $m$ with the ring $R_n$ and the $R_n$-module $M_n$ as in the statement of the proposition. We set $D_{n,m} = R_n/(\mathfrak{b}_m, \mathfrak{m}_R^{(r_m)})$ with its surjection $D_{n,m} \to R/(\mathfrak{m}_S^n, \mathfrak{m}_R^{(m)})$ induced from the surjection $R_n \to R$. We let $B \to D$ be the map induced from the $B$-algebra structure on $R_n$. We note that $\dim_{\mathfrak{m}_{D_{n,m}}/\mathfrak{m}_{D_{n,m}}^2} \leq k$ by our choice of $k$.

We let $L_{n,m} = M_n/\mathfrak{b}_m$. To see that this is a $D$-module, we need to check that the ideal $(\mathfrak{b}_m, \mathfrak{m}_R^{(r_m)})$ in $R_n$ annihilates $M_n/\mathfrak{b}_m$. To see this, note that any $a \in \mathfrak{m}_R$ acts nilpotently on $M_n/\mathfrak{m}_S^n = M_n/(\mathfrak{m}_S, a)$, so $a^*$ annihilates $M_n/(\mathfrak{m}_S, a)$. Since $a_n$ is generated by $h$ elements of the form $(\delta - 1)^{\overline{p}^m - 1}$ (with $\delta$ one of the $h$ standard generators of the group $(\mathbb{Z}_p)^h$ and $p \in \mathfrak{m}_S$, it follows that $a^{\overline{p}^m} M_n \subseteq (\mathfrak{m}_S, a_n) M_n$, and $a^m \overline{p}^m a_n \subseteq (\mathfrak{m}_S^n, a_n) M_n$. The surjection $L_{n,m} \to M/\mathfrak{m}_S^n$ is induced from the surjection $M_n \to M$. The fact that this has kernel $aL_{n,m}$ is implied by (j), and the fact that $L_{n,m}$ is free over $S_\infty/\mathfrak{b}_m$ is implied by (i).

Since there are only finitely many isomorphism classes of patching data of level $m$ for any fixed $m \geq 1$, we see that for each $m \geq 1$, there are infinitely many $n \geq m$ such that the patching data $(D_{n,m}, L_{n,m})$ are all isomorphic. A standard argument gives an infinite subsequence $(n_k)_{k \geq 1}$ of the positive integers such that $(D_{n_k, k}, L_{n_k, k})$ mod $m \cong (D_{n_k, m}, L_{n_k, m})$ for all $k \geq m$. Set $(D_m, L_m) = (D_{n_k, m}, L_{n_k, m})$ for all $m \geq 1$, and fix isomorphisms $\alpha_m : (D_{m+1}, L_{m+1})$ mod $m \cong (D_m, L_m)$ for all $m \geq 1$. Then $(D_m, L_m, \alpha_m)_{m \geq 1}$ forms a projective system and we set

$$R_\infty = \lim_{\substack{\rightarrow m}} D_m, \quad L_m = \lim_{\substack{\rightarrow m}} L_m.$$
Note first that \( \dim_{\mathfrak{F}} \mathfrak{m}_{R_{\infty}}/\mathfrak{m}_{R_{\infty}}^2 \leq k \), so \( R_{\infty} \) is Noetherian. The ring \( R_{\infty} \) comes equipped with a natural surjection \( R_{\infty} \to R \), and a local morphism \( B \to D_{\infty} \). Let \( \mathfrak{p}_{\infty} \) be the pullback of \( p \) to \( R_{\infty} \). By the construction of \( R_{\infty} \) and \( \mathfrak{f} \), we see that \( \mathfrak{p}_{\infty} \) pulls back to \( \mathfrak{p}_B \) in \( B \). By \( \mathfrak{g} \) and the construction of \( R_{\infty} \), the finite cardinality \( \mathcal{O} \)-modules \( \mathfrak{p}_{\infty}/(\mathfrak{p}_{\infty}, \mathfrak{p}_B, \mathfrak{w}^n) \) have order bounded independently of \( n \). This implies that map on localizations and completions

\[
B_{\mathfrak{p}_B}^\wedge \to (R_{\infty})_{\mathfrak{p}_{\infty}}^\wedge
\]

is surjective (both have residue field \( E \)).

Part (iii) (in the definition of patching data implies that \( M_\infty \) is a finite free \( S_\infty \)-module and there is an isomorphism \( M_\infty/\mathfrak{a} \cong M \) of \( R_\infty \)-modules, with the \( R_\infty \)-module structure on \( M \) coming from the surjection \( R_\infty \to R \). This together with the fact that \( \mathfrak{p} \) contains \( z_1, \ldots, z_j \) implies we can find \( M_\infty \)-regular sequence of length \( h + j \) in \( \mathfrak{p}_{\infty} \) and

\[
\text{depth}_{(R_{\infty})_{\mathfrak{p}_{\infty}}^\wedge} (M_{\infty})_{\mathfrak{p}_{\infty}}^\wedge \geq \text{depth}_{\mathfrak{p}_{\infty}} (\mathfrak{p}_{\infty}, M_{\infty}) \geq h_j.
\]

On the other hand, assumption (d) and the surjection (7) implies \( \dim(R_{\infty})_{\mathfrak{p}_{\infty}}^\wedge = h + j \), so we have

\[
\text{depth}_{(R_{\infty})_{\mathfrak{p}_{\infty}}^\wedge} (M_{\infty})_{\mathfrak{p}_{\infty}}^\wedge = h + j = \dim(R_{\infty})_{\mathfrak{p}_{\infty}}^\wedge.
\]

The Auslander-Buchsbaum formula then implies that \( (M_{\infty})_{\mathfrak{p}_{\infty}}^\wedge \) is a finite free \( (R_{\infty})_{\mathfrak{p}_{\infty}}^\wedge \)-module. Then \( (M_{\infty})_{\mathfrak{p}_{\infty}}^\wedge/\mathfrak{a} \) is a finite free \( (R_{\infty})_{\mathfrak{p}_{\infty}}^\wedge/\mathfrak{a} \)-module. But \( (M_{\infty})_{\mathfrak{p}_{\infty}}^\wedge/\mathfrak{a} \cong M_{\mathfrak{p}}^\wedge \) and the action of \( (R_{\infty})_{\mathfrak{p}_{\infty}}^\wedge/\mathfrak{a} \) on it is via \( R_{\mathfrak{p}}^\wedge \). This proves the proposition. \( \square \)

3.2. Galois cohomology and Taylor–Wiles primes. In this subsection we construct the Taylor–Wiles systems necessary to apply 3.1.1. We follow [Kis04, §6] closely, and many of the lemmas are taken directly (in statement or proof or both) directly from there.

Let \( F \) be a totally real field and \( S \) denotes a finite set of places of \( F \) containing all those above \( p \) and all the infinite places. We let \( S_f = S \setminus \{ \mathfrak{v} | \mathfrak{v} \} \). In this subsection we fix a continuous

\[
\rho : G_{F,S} \to \text{GL}_2(\mathcal{O}),
\]

and set \( \overline{\mathfrak{p}} = \rho \otimes_\mathcal{O} \mathbb{F} \) and \( \rho_E = \rho \otimes_\mathcal{O} E \). We assume that there is a regular algebraic cuspidal automorphic representation \( \pi \) of \( \text{GL}_2(\mathbb{A}_F) \) such that \( \rho_E \cong \rho_\pi \) (see 2.2.1). We also assume that \( E \) contains the eigenvalues of all elements in the image of \( \overline{\mathfrak{p}} \).

If \( N \) is any \( \mathcal{O} \)-module, and \( A \) is an \( \mathcal{O} \)-algebra, we set \( N_A = N \otimes A \). In the case that \( A = \mathcal{O}/\mathfrak{w}^n \) we write \( N_n = N/\mathfrak{w}^n N \). We let \( \text{ad} = \text{ad}(\rho) \), \( \text{ad}^0 \) be its trace zero subspace, and \( \mathfrak{z} \) be the submodule of scalar endomorphisms. Note that \( \text{ad}_{E} = \text{ad}(\rho_E) \), and \( \text{ad}_{\mathfrak{z}} = \text{ad}(\rho \otimes \mathfrak{w}^n) \) for any \( n \geq 1 \).

If \( W \) is a finite \( \mathcal{O} \)-module, we let \( W^\vee = \text{Hom}_{\mathcal{O}}(W, E/\mathcal{O}) \) be its Pontryagin dual. If \( W \) is a finite \( E \)-vector space we let \( W^\vee = \text{Hom}_{E}(W, E) \) be its \( E \)-linear dual. Given \( W \) either a finite \( \mathcal{O} \)-module or finite \( E \)-vector space with a continuous \( \mathcal{O} \)-, resp. \( E \)-, linear \( G_{F,S} \)-action, if \( Q \) is a finite (possibly empty) set of primes disjoint from \( S \), we define a Selmer group

\[
H_{S_Q}^1(G_{F,S\cup Q}, W) = \{ \gamma \in H^1(G_{F,S\cup Q}, W) \mid \text{res}_v(\gamma) = 0 \text{ for all } v \in S_f \},
\]

which has dual Selmer group

\[
H_{S_Q}^1(G_{F,S\cup Q}, W^\vee(1)) = \{ \gamma \in H^1(G_{F,S\cup Q}, W^\vee(1)) \mid \text{res}_v(\gamma) = 0 \text{ for all } v \in Q \cup \{ \mathfrak{v} \} \}.
\]

For any \( n \geq 1 \), the trace pairing \( (X, Y) = \text{tr}(XY) \) on \( \text{ad}_n \) is perfect, so combining it with the isomorphism \( \mathcal{O}/\mathfrak{w}^n \cong \mathfrak{w}^{-n}\mathcal{O}/\mathcal{O} \), we have an \( \mathcal{O}[G_F] \)-module isomorphism between \( \text{ad}_n \) and its Pontryagin dual. The trace pairing induces a perfect pairing between \( \text{ad}_n^0 \) and \( \text{ad}_n/\mathfrak{z}n \), so \( \text{ad}_n/\mathfrak{z}n \) is isomorphic to the Pontryagin dual of \( \text{ad}_n^0 \) and vice-versa. Similarly, the trace pairing induces an \( E[G_F] \)-module isomorphism between the \( E \)-linear dual of \( \text{ad}_n^0 \) and \( \text{ad}_E/\mathfrak{z}E \cong \text{ad}_n^0 \).

3.2.1. Let \( G \) be either \( G_{F,S\cup Q} \) or \( G_{\mathfrak{v}} \) for some place \( \mathfrak{v} \) of \( F \), and let \( W \) be a finite \( \mathcal{O} \)-module with continuous \( \mathcal{O} \)-linear \( G \) action. Then for any \( n \geq 1 \), the minimal number of generators of the \( \mathcal{O} \)-module \( H^1(G, W_n) \) is bounded independently of \( n \). Moreover, the size of \( H^1(G, W_n) \) is bounded independently of \( n \) if and only if \( H^1(G, W_E) = 0 \). Both facts follows from the long exact sequence associated to

\[
0 \to W \to \mathfrak{w}^n W \to W_n \to 0
\]
Lemma 3.2.2. Let $M_n$ be the fixed field of the kernel of $G_F \to \text{Aut}_\mathcal{O}(\text{ad}_n/\mathfrak{f}_n(1))$. The finite groups $H^1(\text{Gal}(M_n/F), \text{ad}_n/\mathfrak{f}_n(1))$ have order bounded independently of $n$.

Proof. Let $M_\infty$ be the fixed field of the kernel of the map $G_F \to \text{Aut}_\mathcal{O}(\text{ad}/\mathfrak{f}(1))$. By inflation it suffices to show $H^1(\text{Gal}(M_\infty/F), \text{ad}_n/\mathfrak{f}_n(1))$ has order bounded independently of $n$. For this it suffices to show $H^1(\text{Gal}(M_\infty/F), \text{ad}_E/\mathfrak{f}_E(1)) = 0$, and this can be shown exactly as in [Ser71, Cor]

Lemma 3.2.3. $H^0(G_F, \text{ad}_E^0) = H^0(G_F, \text{ad}_E^0(1)) = 0$.

Proof. The first claim follows from the assumption that $\rho_E$ is absolutely irreducible. For the second, any $X \in H^0(G_F, \text{ad}_E^0(1))$ defines a homomorphism $V_{\rho_E} \to V_{\rho_E}(1)$ of $E[G_F]$-modules. By Chebotarev density, we can find some $v \notin S$, such that the trace of $\text{Frob}_v$ on $V_{\rho_E}$ is nonzero. The trace of such an element on $V_{\rho_E}$ and $V_{\rho_E}(1)$ are distinct, so $X$ cannot be an isomorphism. The absolute irreducibility of $V_{\rho_E}$ then implies $X = 0$.

Lemma 3.2.4. Let $v$ be a finite place of $F$ not contained in $S$. If $\text{Nm}(v) \equiv 1 \pmod{p^n}$ and $\bar{\mathfrak{p}}(\text{Frob}_v)$ has distinct eigenvalues, then $|H^1(G_v, \text{ad}_n/\mathfrak{f}_n(1))| = |F|^n$

Proof. By local Tate duality and local Euler characteristic

$$|H^1(G_v, \text{ad}_n/\mathfrak{f}_n(1))| = \frac{|F|^{3n}}{|H^0(G_v, \text{ad}_n/\mathfrak{f}_n(1))||H^0(G_v, \text{ad}_n^0)|}.$$

Since $\text{Nm}(v) \equiv 1 \pmod{p^n}$, $\text{ad}_n/\mathfrak{f}_n(1) \cong \text{ad}_n/\mathfrak{f}_n$ as $(O/\mathfrak{p}^n)[G_v]$-modules. Using the fact that $\bar{\mathfrak{p}}(\text{Frob}_v)$ has distinct $F$-rational eigenvalues, we can diagonalize $\rho(\text{Frob}_v)$ over $O$, and it is easily checked in such a basis that

$$|H^0(G_v, \text{ad}_n^0)| = |H^0(G_v, \text{ad}_n/\mathfrak{f}_n)| = |F|^n.$$ 

Lemma 3.2.5. $\dim_E H^1_{S_0}(G_{F,S}, \text{ad}_E^0) = \dim_E H^1(G_{F,S}, \text{ad}_E^0(1)) - [F : \mathbb{Q}] \sum_{v \in S} \dim_E H^0(G_v, \text{ad}_E^0)$.

Proof. Since $H^1(G_v, \text{ad}_E^0(1)) = 0$ for all $v|\infty$, we have $H^1_{S_0}(G_{F,S}, \text{ad}_E^0(1)) = H^1(G_{F,S}, \text{ad}_E^0(1))$. The Greenberg–Wiles formula yields

$$\dim_E H^1_{S_0}(G_{F,S}, \text{ad}_E^0) = \dim_E H^1(G_{F,S}, \text{ad}_E^0(1)) + \dim_E H^0(G_{F,S}, \text{ad}_E^0) - \dim_E H^0(G_{F,S}, \text{ad}_E^0(1))$$

$$- \sum_{v \in S} \dim_E H^0(G_v, \text{ad}_E^0).$$

The lemma follows from 3.2.3 and the fact that $\dim_E H^0(G_v, \text{ad}_E^0) = 1$ for each $v|\infty$, since $\rho_E$ is totally odd.

The following two lemmas are almost identical in both statement and proof to [Kis04, Proposition 6.4 and 6.9].

Proposition 3.2.6. Assume the following.

(a) $\rho(G_L)$ is nonabelian for any CM extension $L/F$.

(b) $\bar{\mathfrak{p}}|_{G_F(G_v)}$ has nonscalar semisimplification.

Let $Q_n$ denote the set of finite places $v$ of $F$ disjoint from $S$ such that

- $\text{Nm}(v) \equiv 1 \pmod{p^n}$,
- $\text{Frob}_v$ has distinct eigenvalues on $V_{\rho}$.

Then for each $n \geq 1$, the kernel of the map

$$H^1(G_{F,S}, \text{ad}_n/\mathfrak{f}_n(1)) \rightarrow \prod_{v \in Q_n} H^1(G_v, \text{ad}_n/\mathfrak{f}_n(1)).$$

has order bounded independently of $n$.
Proof. Since the minimal number of generators of $H^1(G_{F,S}, \text{ad}_n/\mathfrak{z}_n(1))$ as an $\mathcal{O}$-module is bounded independently of $n$, the same is true for the kernels of (8) since $\mathcal{O}$ is a discrete valuation ring. So it suffices to show that there is a constant $c$ such that for any $\gamma$ in the kernel of (8), $\varpi^c \gamma = 0$. Let $M_n$ be the fixed field of the kernel of $G_{F,s} \to \text{Aut}_\mathcal{O}(\text{ad}_n/\mathfrak{z}_n(1))$. By 3.2.2, it suffices to show that there is a constant $c$, independent of $n$ and $\gamma$, such that the image of $\gamma$ in

$$H^1(G_{M_n}, \text{ad}_n/\mathfrak{z}_n(1))^{\text{Gal}(M_n/F)} = \text{Hom}_{\text{Gal}(M_n/F)}(G_{M_n}, \text{ad}_n/\mathfrak{z}_n(1))$$

is annihilated by $\varpi^c$.

Let $F_\infty = \bigcup_{n \geq 1} F(\zeta_p^n)$. Since $F_\infty/F(\zeta_p)$ is pro-$p$, the restriction $\mathfrak{p}|_{G_{F(\zeta_p)}}$ has nonscalar semisimplification if and only if the restriction $\mathfrak{p}|_{G_{F_\infty}}$ has nonscalar semisimplification. So by assumption (b), there is $\sigma_0 \in G_{F_\infty}$ such that $\mathfrak{p}(\sigma_0)$ has distinct eigenvalues, which we fix.

If $\text{ad}_E^{\mathfrak{p} \otimes E} E$ was a reducible $G_F$-representation, it would have a one-dimensional $G_F$-stable subquotient. By the trace pairing, it would have a one-dimensional subrepresentation, and there would be an abelian extension $L/F$ and a nonscalar endomorphism of $V_{\rho_E}$ that commutes with $\rho(G_L)$, which implies that $\rho_E|_{G_L}$ is reducible. Since $\rho_E$ is absolutely irreducible, Clifford theory implies that there is a quadratic subextension $L/F'$ of $L/F$ such that $\rho|_{G_{L/F'}}$ is reducible. Since $\rho_E \cong \rho_\pi$ for a regular algebraic cuspidal automorphic representation $\pi$ of $GL_2(\mathbb{A}_F)$, the extension $L/F$ must be CM (apply the argument of [Tay, Proposition 3.1] to the base change of $\pi$ to $L$). Thus, assumption (a) implies that $\rho_E|_{G_{L/F'}} \cong \text{ad}_G^{\mathfrak{p}}$ is absolutely irreducible.

This implies that there is a constant $c$, depending only on $\rho(G_F)$, such that for any $n \geq 1$ and any $\mathcal{O}[G_F]$-submodule $W$ of $\text{ad}_n/\mathfrak{z}_n(1)$ whose image in $\text{ad}_n/\mathfrak{z}_n(1)$ is nonzero, satisfies $\varpi^c \text{ad}_n/\mathfrak{z}_n(1) \subseteq W$. This in turn implies that for any $n \geq 1$, if $W$ is a $\mathcal{O}[G_F]$-stable submodule of $\text{ad}_n/\mathfrak{z}_n(1)$ such that $W \not\subseteq \varpi^{k+1} \text{ad}_n/\mathfrak{z}_n(1)$, then $\varpi^{k+1} \text{ad}_n/\mathfrak{z}_n(1) \subseteq W$.

Let $X$ be the $\mathcal{O}$-span of $\kappa(G_{M_n})$ in $\text{ad}_n/\mathfrak{z}_n(1)$, and choose $k$ such that $X \not\subseteq \varpi^k \text{ad}_n/\mathfrak{z}_n(1)$. We will be finished if we show $k \geq n - c$. By above, we know that $\varpi^{k+c} \text{ad}_n/\mathfrak{z}_n(1) \subseteq X$. By Chebotarev density, for any $\tau \in G_{M_n}$, we can find a prime $v$ of $F$ disjoint from $S$ such that $\left(\sigma_0 \tau \text{Frob}_v^{-1}\right) \in \ker(\kappa|_{G_{F_n}})$. By our assumption on $\sigma_0$, any such $v$ is in $Q_n$. Since $\gamma$ is in the kernel of (8), we have

$$\kappa(\tau \sigma_0) \in (\sigma - 1) \text{ad}_n/\mathfrak{z}_n(1) = (\sigma_0 - 1) \text{ad}_n/\mathfrak{z}_n(1).$$

for all $\tau \in G_{M_n}$. Taking $\tau = 1$, we have $\kappa(\sigma_0) \in (\sigma_0 - 1) \text{ad}_n/\mathfrak{z}_n(1)$. For any $\tau \in G_{M_n}$, we have $\kappa(\tau \sigma_0) = \kappa(\sigma_0 + \kappa(\tau)$, so we conclude that $\kappa(G_{M_n}) \subseteq (\sigma_0 - 1) \text{ad}_n/\mathfrak{z}_n(1)$, hence

$$\varpi^{k+c} \text{ad}_n/\mathfrak{z}_n(1) \subseteq X \subseteq (\sigma_0 - 1) \text{ad}_n/\mathfrak{z}_n(1).$$

As straightforward computation in a basis that diagonalizes $\mathfrak{p}(\sigma_0)$ shows that $\varpi^{n-1} \text{ad}_n/\mathfrak{z}_n(1) \not\subseteq (\sigma_0 - 1) \text{ad}_n/\mathfrak{z}_n(1)$, so $k \geq n - c$. \qed

Proposition 3.2.7. Assume the following.

(a) $p > 2$,
(b) $\mathfrak{p}$ is absolutely irreducible,
(c) there is a CM extension $L/F$ with $L \not\subseteq F(\zeta_p)$ such that $\pi$ has CM by $L$.

Let $Q_n$ denote the set of finite places $v$ of $F$ disjoint from $S$ such that

- $\text{Nm}(v) \equiv 1 \pmod{p^n}$,
- $\mathfrak{p}(\text{Frob}_v)$ has distinct eigenvalues.

Then for every $n \geq 1$, the kernel of the map

$$H^1(G_{F,S}, \text{ad}_n/\mathfrak{z}_n(1)) \to \prod_{v \in Q_n} H^1(G_v, \text{ad}_n/\mathfrak{z}_n(1)).$$

has order bounded independently of $n$.

Proof. Since $\rho_E$ is absolutely irreducible, assumption (b) implies that, extending scalars if necessary, we can write $\rho_E = \text{Ind}_{G_L}^G \chi$ for a character $\chi : G_L \to E^\times$. It can then be checked that there is an $\mathcal{O}[G_F]$-module isomorphism $\text{ad}_n/\mathfrak{z}_n \cong \mathcal{O}[\delta_L] \oplus Y$, where $\delta_L$ is the quadratic character of $L/F$, and $Y \otimes \mathcal{O} E = \text{Ind}_{G_L}^G (\chi(\chi')^{-1})$ where $\chi'$ is the conjugate of $\chi$ by the nontrivial element of $\text{Gal}(L/F)$. Let $W$ denote one of $\mathcal{O}[\delta_L]$ or $Y$, and let $W_n = W/\varpi^n$ for every $n \geq 1$. We want to show

$$H^1(G_{F,S}, W_n(1)) \to \prod_{v \in Q_n} H^1(G_v, W_n(1)).$$

...
has order bounded independently of \( n \geq 1 \).

If \( W = \mathcal{O}(\delta_L) \), then \( W \otimes_{\mathcal{O}} E = E(\delta_L) \) is clearly absolutely irreducible. Let \( F_{\infty} = \bigcup_{n \geq 1} F(\zeta_p^n) \). The assumption that \( \mathfrak{p} \) is absolutely irreducible implies that \( \rho(G_{L_{\infty}}) \) is noncentral, since \( L_{\infty}/F \) is abelian. Any element of \( \mathfrak{p}(G_L) \) is semisimple, hence there is \( \sigma_0 \in G_{L_{\infty}} \) with distinct eigenvalues. We then argue as in 3.2.6, using the element \( \sigma_0 \), as \( (\sigma_0 - 1)(\mathcal{O}/\mathfrak{p}^n)(\delta_L \varepsilon) = 0 \) for any \( n \geq 1 \).

We now consider \( W = Y \). Since the image of \( \rho(G_F) \) in \( \text{PGL}_2(\mathcal{O}) \) is infinite, the character \( \chi(\chi')^{-1} \) is not quadratic. This implies that \( Y \otimes_{\mathcal{O}} E \) is also absolutely irreducible. Since \( p > 2 \), any element of \( G_F \) has distinct eigenvalues under \( Y \). It follows that there is an \( E \)-line in \( Y \otimes_{\mathcal{O}} E \) fixed by \( \sigma_0 \). This and the fact that \( \varepsilon(\sigma_0) = 1 \), imply there is an element in \( Y(\mathcal{O}) \otimes_{\mathcal{O}} E \) on which \( \sigma_0 \) acts trivially. This implies that \( \varpi^{n-1} Y_n(1) \not\subseteq (\sigma_0 - 1)Y_n(1) \), and we can again argue as in 3.2.6.

We now recall [Kis04, Lemma 6.5].

**Lemma 3.2.8.** Let \( W \) be a \( \mathcal{O} \)-module generated by \( \mathfrak{p} \) elements. Assume there is a family \( (W_i)_{i \in I} \) of finite length \( \mathcal{O} \)-modules and a \( \mathcal{O} \)-module map \( W \to \prod_{i \in I} W_i \). Then there are \( h \) distinct indices \( i_1, \ldots, i_h \in I \) such that the image of \( W \) in \( \prod_{i=1}^h W_i \), is isomorphic to the image of \( W \) in \( \prod_{i \in I} W_i \).

**Proof.** We may replace \( W \) with its image in \( \prod_{i \in I} W_i \), and this is [Kis04, Lemma 6.5], \( \square \)

**Proposition 3.2.9.** Assume the following.

(a) \( \mathfrak{p}|G_{F(\mathcal{O})} \) has nonscalar semisimplification.

(b) If \( \pi \) has complex multiplication by a quadratic CM extension \( L/F \), then we assume

- \( p > 2 \),
- \( \mathfrak{p} \) is absolutely irreducible,
- \( L \not\subseteq F(\zeta_p) \).

Then for all \( n \geq 1 \) there is a set of finite places \( Q_n \) of \( F \), disjoint from \( S \) and satisfying the following properties.

1. \( |Q_n| = \dim_E H^1(G_{F,S}, \text{ad}_E^0(1)) \).
2. \( \text{Nm}(v) \equiv 1 \pmod{p^n} \) for all \( v \in Q_n \).
3. \( \mathfrak{p}|\text{Frob}_v \) has distinct eigenvalues for all \( v \in Q_n \).
4. For all \( n \geq 1 \), there is an \( \mathcal{O} \)-module map

\[ (\mathcal{O}/\mathfrak{p}^n)^r \to H^1_{S,Q_n}(G_{F,S}, \text{ad}_E^0) \]

with kernel and cokernel of size bounded independently of \( n \), where

\[ r = \dim_E H^1(G_{F,S}, \text{ad}_E^0(1)) - [F:Q] - \sum_{v \in S} \dim_E H^0(G_v, \text{ad}^0). \]

**Proof.** Fix a direct summand \( W \) of \( H^1(G_{F,S}, \text{ad}/3(1)) \) that is free of rank \( \dim_E H^1(G_{F,S}, \text{ad}_E^0(1)) \) over \( \mathcal{O} \). Let \( Q_n \) be as in 3.2.6 and 3.2.7. By our assumptions (a) and (b), the conclusions of 3.2.6 and 3.2.6 hold. By 3.2.8, there is a subset \( Q_n \subset Q_n \) of cardinality \( \dim_E H^1(G_{F,S}, \text{ad}_E^0(1)) \), such that the image of \( W \) in \( \prod_{v \in Q_n} H^1(G_v, \text{ad}_v/3(1)) \) is isomorphic to the image of \( W \) in \( \prod_{v \in Q_n} H^1(G_v, \text{ad}_v/3(1)) \). Thus for each \( n \geq 1 \), we have a set of places \( Q_n \), disjoint from \( S \) and satisfying parts 1., 2., and 3. of the lemma, as well as that the maps

\[ W/\mathfrak{p}^n \to H^1(G_{F,S}, \text{ad}_v/3(1)) \quad \text{and} \quad W/\mathfrak{p}^n \to \prod_{v \in Q_n} H^1(G_v, \text{ad}_v/3(1)) \]

have kernel of size bounded independently of \( n \). The first also has cokernel of size bounded independently of \( n \), and so does the second by 3.2.4. We deduce that the map

\[ H^1(G_{F,S}, \text{ad}_v/3(1)) \to \prod_{v \in Q_n} H^1(G_v, \text{ad}_v/3(1)) \]

has kernel and cokernel of size bounded independently of \( n \).
To see part 4., first note that the $O$-rank of $H^1_{\mathcal{S}_N}(G_{F,S}, \text{ad}^0)$ is equal to
\[ r = \dim_E H^1(G_{F,S}, \text{ad}^0_E(1)) - [F : \mathbb{Q}] - \sum_{v \in S_f} \dim_E H^0(G_v, \text{ad}^0) \]
by 3.2.5. Then, since the natural map
\[ H^1_{\mathcal{S}_N}(G_{F,S}, \text{ad}^0)/\mathbb{Z}^n \longrightarrow H^1_{\mathcal{S}_N}(G_{F,S}, \text{ad}^0_n) \]
has kernel and cokernel of size bounded independently of $n$, it suffices to show that the natural injection
\[ H_N(G_{F,S}, \text{ad}^0_n) \longrightarrow H^1_{\mathcal{S}_{Q_n}}(G_{F,S\cup Q_n}, \text{ad}^0_n) \]
has cokernel of size bounded independently of $n$.

Consider the map
\[ H^1(G_{F,S\cup Q_n}, \text{ad}^0_n/\mathfrak{m}_n(1)) \longrightarrow \prod_{v \in Q_n} H^1(G_v, \text{ad}^0_n/\mathfrak{m}_n(1)). \]
Since (9) factors through (11), we deduce that the cokernel of (11) has size bounded independently of $n$. Since the kernel of (11) factors through $H^1(G_{v,S}, \text{ad}^0_n/\mathfrak{m}_n(1))$, the kernel of (11) also has size bounded independently of $n$. For each $v$, since $H^1(G_v, \text{ad}^0_E/\mathfrak{m}_E(1)) = 0$, we deduce that the kernel and cokernel of
\[ H^1(G_{F,S\cup Q_n}, \text{ad}^0_n/\mathfrak{m}_n(1)) \longrightarrow \prod_{v \in Q_n \cup \{w|\infty\}} H^1(G_v, \text{ad}^0_n/\mathfrak{m}_n(1)) \]
has size bounded independently of $n$. Taking Pontryagin duals and applying Tate local duality to the $H^1(G_v, \text{ad}^0_n/\mathfrak{m}_n(1))$, we see that the natural map
\[ \prod_{v \in Q_n \cup \{w|\infty\}} H^1(G_v, \text{ad}^0_n) \longrightarrow H^1(G_{F,S\cup Q_n}, \text{ad}^0_n/\mathfrak{m}_n(1))^\vee \]
has kernel and cokernel bounded independently of $n$. Using the Poitou–Tate exact sequence, (12) fits into the exact sequence
\[ H^1_{\mathcal{S}_{Q_n}}(G_{F,S\cup Q_n}, \text{ad}^0_n) \longrightarrow \prod_{v \in Q_n \cup \{w|\infty\}} H^1(G_v, \text{ad}^0_n) \longrightarrow H^1(G_{F,S\cup Q_n}, \text{ad}^0_n/\mathfrak{m}_n(1))^\vee. \]
From this we see that the image of $H^1_{\mathcal{S}_{Q_n}}(G_{F,S\cup Q_n}, \text{ad}^0_n) \rightarrow \prod_{v \in Q_n} H^1(G_v, \text{ad}^0_n)$ has size bounded independently of $n$. Since the kernel of this map is contained in the image of (10), we deduce that the cokernel of (10) has size bounded independently of $n$. \[\square\]

Remark 3.2.10. We have shown above that the Taylor–Wiles primes necessary to construct rings $R_n$ and $M_n$ satisfying (most of) the assumptions of 3.1.1 exist in most residually reducible cases. However, the author is unable to adapt the patching argument to work in these cases. In [SW00] and [Tho14a], the authors there construct modules of cusp forms $M_n$, deformation rings $R_n$, and pseudodeformation rings $P_n$, and they must patch the triple of objects because $M_n$ is a module over $P_n$ and not over $R_n$. One of the subtleties is to control the relationship between $P_n$ and $R_n$ when patching.

In [SW00], this is done by a two step patching argument. There are three different parameters that one can vary in the patching argument here, one for the choice of Taylor–Wiles data (the $n$ in 3.1.1), one for the level of the patching variables (the $m$ in the proof of 3.1.1), and another for the power of the uniformizer of the local field in question (these are $N$, $a$, and $c$, respectively, in [SW00, §5]). For every fixed level of the patching variables, they can control the relation between the deformation and pseudodeformation rings (see [SW00, Lemma 5.2]), so they first patch along the auxiliary data and power of the uniformizer, then localize and complete at the fixed dimension one prime ideal, and then take a limit of these patched objects over the level of the patching variables (see [SW00, pg. 83–85]). This technique is unavailable in our current context because we are patching along a dimension one prime ideal of characteristic zero, and the Taylor–Wiles primes $Q_n$ constructed above only have the relevant properties modulo $p^m$ with $m \leq n$. We only succeed in gluing the data if we keep the powers of the uniformizer growing no faster than the level of the patching variables. This is really just the simple fact that if $\Delta$ is a finite $p$-group, then $O[\Delta][1/p]$ is étale.

In [Tho14a], Thorne uses the assumption that the residual representation is Schur to show the map between the pseudodeformation and deformation rings is finite (see [Tho14a, Proposition 3.29]), and he then uses a Fitting ideal argument (see [Tho14a, Lemma 4.22]) in the patching. This does not hold in our
context because extensions of two characters in characteristic $p$ give positive dimensional families of reducible representations lying over the same pseudorepresentation. It also does not help us to base change to a CM field and use the semisimple representation, because if a two dimensional representation of a totally real field is absolutely reducible, then its restriction to a CM field is never Schur.

### 3.3. Adjoint Selmer groups.

We are now ready to prove Theorem A. We recall the set up.

Let $p$ be a prime and let $E$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $F$. Let $F$ be a totally real field, and let $S$ be a finite set of places of $F$ containing all places above $p$ and all places above $\infty$. Let $G_{F,S}$ be the Galois group of the maximal extension of $F$ unramified outside of $S$ in some fixed algebraic closure. Let

$$\rho : G_{F,S} \to \text{GL}_2(E)$$

be a continuous representation, and let $V_\rho$ denote the representation space of $\rho$. Let $\text{ad}(\rho)$ be the adjoint representation on $\text{Hom}_E(V_\rho, V_\rho)$. We let $\text{ad}^0(\rho)$ be the representation of $G_{F,S}$ on the trace zero subspace of $\text{Hom}_E(V_\rho, V_\rho)$. We recall that if $W$ is a $\mathbb{Q}_p$-vector space with a continuous $\mathbb{Q}_p$-linear action of $G_{F,S}$, then the geometric Bloch–Kato Selmer group for $W$ is

$$H^1_g(G_{F,S}, W) := \ker \left( H^1(G_{F,S}, W) \to \prod_{v \mid p} H^1(G_v, B_{\text{dR}} \otimes \mathbb{Q}_p, W) \right)$$

Finally, we recall that we can choose a $G_{F,S}$-stable $\mathcal{O}$-lattice in $V_\rho$, so after conjugation we may assume that $\rho$ takes values in $\text{GL}_2(\mathcal{O})$. The semisimplification of its reduction modulo the maximal ideal of $\mathcal{O}$ does not depend on the choice of lattice, and we denote it by $\overline{\rho} : G_{F,S} \to \text{GL}_2(\mathbb{F})$.

**Theorem 3.3.1.** Assume there is a finite totally real extension $F'/F$, a regular algebraic cuspidal automorphic representation $\pi$ of $\text{GL}_2(\mathbb{A}_{F'})$, and an isomorphism $\iota : \mathbb{C} \sim \mathbb{C}_p$ such that the following hold.

1. $\rho|_{G_{F'}} \cong \rho_{\pi,1}$.
2. $\overline{\rho}|_{G_{F'}}$ is absolutely irreducible.
3. If $\pi$ has complex multiplication by a quadratic extension $L/F'$, we assume $p > 2$ and $L \not\subseteq F'(\zeta_p)$.

Then $H^1_g(G_{F,S}, \text{ad}(\rho)) = H^1_g(G_{F,S}, \text{ad}^0(\rho)) = 0$.

**Proof.** The fact that $H^1_g(G_{F,S}, \text{ad}(\rho)) = 0$ implies $H^1_g(G_{F,S}, \text{ad}(\rho)) = 0$ is standard. Indeed, $\text{ad}(\rho) \cong E \oplus \text{ad}^0(\rho)$, so

$$H^1_g(G_{F,S}, \text{ad}(\rho)) \cong H^1_g(G_{F,S}, \text{ad}^0(\rho)) \oplus H^1_g(G_{F,S}, E).$$

For each $v | p$, the local group $H^1_g(G_v, E) = \ker (H^1(G_v, E) \to H^1(G_v, B_{\text{dR}} \otimes \mathbb{Q}_p, E))$ is the one dimensional $E$-subspace of $\text{Hom}(G_v, E)$ corresponding to the unramified extension. By class field theory, $H^1_g(G_{F,S}, E) = 0$.

To prove $H^1_g(G_{F,S}, \text{ad}^0(\rho)) = 0$, we first note that we are free to enlarge $S$. We are also free to replace $F$ with any finite extension $L/F$ such that $\rho|_{G_L}$ is absolutely irreducible. Indeed, the absolute irreducibility of $\rho|_{G_L}$ implies that $(\text{ad}^0(\rho))|_{G_L} = 0$, so the restriction $H^1(G_{F,S}, \text{ad}(\rho)) \to H^1(G_{L,S_L}, \text{ad}(\rho|_{G_L}))$ is injective by the inflation-restriction exact sequence, where we have denoted by $S_L$ the set of places of $L$ above $S$. The commutativity of

$$\begin{array}{ccc}
H^1(G_{F,S}, \text{ad}(\rho)) & \longrightarrow & \prod_{v \in S} H^1(G_v, B_{\text{dR}} \otimes \mathbb{Q}_p, \text{ad}(\rho)) \\
\downarrow & & \downarrow \\
H^1(G_{L,S_L}, \text{ad}(\rho)) & \longrightarrow & \prod_{w \in S_L} H^1(G_w, B_{\text{dR}} \otimes \mathbb{Q}_p, \text{ad}(\rho|_{G_L}))
\end{array}$$

implies that $H^1_g(G_{F,S}, \text{ad}(\rho))$ has image in $H^1_g(G_{F,S}, \text{ad}(\rho|_{G_L}))$, so the former is trivial if the latter is. Thus, we can replace $F$ with $F'$ as in the statement of the theorem. Using Langlands cyclic base change we may further replace $F$ by a totally real solvable extension so that $F$, $\rho$, $\overline{\rho}$, and $\pi$ satisfy:

- $[F : \mathbb{Q}]$ is even;
- $\rho \cong \rho_{\pi,1}$;
- $\overline{\rho}$ is absolutely irreducible;
- if $\pi$ has complex multiplication by a quadratic CM extension $L/F$, then $p > 2$ and $L \not\subseteq F(\zeta_p)$;
- $\pi_v$ has Iwahori fixed vectors for every $v | p$. 

Enlarging $E$ if necessary, we assume that $E$ contains all embedding of $F$ into $\mathbb{Q}_p$, and such that $F$ contains the eigenvalues of all elements in the image of $\mathfrak{p}$. We identify $J_F$ with the set of embedding of $F$ into $\mathbb{R}$ using our isomorphism $\iota : \mathbb{C} \cong \mathbb{Q}_p$. If $\tau \in J_F$ gives rise to the infinite place $v$, we let $k_\tau, w_\tau \in \mathbb{Z}$ be such that $\tau_v$ is a discrete series representation with lowest weight $k_\tau - 1$ and central character $z \mapsto \text{sgn}(z)^{k_\tau} |z|^{2-k_\tau-2w_\tau}$. Then let $\kappa = ((k_\tau), (w_\tau)) \in Z^{J_F} \times Z^{J_F}$ is an algebraic weight. Let $\theta_C$ is the central character of $\pi$ and define $\theta : F^\times \backslash (\mathbb{A}_F^\infty)^\times \rightarrow O^\times$ by

$$\theta(z) = \iota(\theta_C(z) \prod_{\tau \in J_F} \tau(z_\infty)^{k_\tau + 2w_\tau - 2} \tau(z_p)^{2-k_\tau-2w_\tau}).$$

Let $D$ be the quaternion algebra with centre $F$ that is ramified at all infinite places and split at all finite places. We fix a maximal order $\mathcal{O}_D$ in $D$, and an isomorphism $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z} \cong M_{2 \times 2}(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z})$. Then $\pi$ is generated by an eigenfunction $f \in S_{\kappa, \theta}(U, E)$ under the Jacquet–Langlands correspondence, with $U$ some open compact subgroup of $\GL_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z})$ such that $\mathcal{U}_v$ contains the Iwahori subgroup for each $v | p$. By enlarging $S$ and shrinking $U_w$ at some $w \not| p$, we can further assume that

$$(U(K^\infty_F)^\times \cap t^{-1}D^\times t)/F^\times = 1$$

for any $t \in (D \otimes_F \mathbb{A}_F^\infty)^\times$. We let $m$ denote the maximal ideal in $T_{\kappa, \theta}(U)$ defined by $\mathfrak{p}$, and let $S_{\kappa, \theta}(U)_m$ denote the localization of $S_{\kappa, \theta}(U)$ at $m$.

Let $S_f = S - \{v|\infty\}$ and let $\psi = \theta^{-1}$. We consider the global deformation datum $S = (S, S_f, \mathfrak{p}, \psi, (\mathcal{R}_v)_{v \in S_f})$ where

- if $v \not| p$, then $\mathcal{R}_v$ is $R^\square_{\kappa, \psi}$ modulo its $p$-torsion;
- if $v | p$, then $\mathcal{R}_v = R^\square_{\kappa, \psi}(\tau_p, \nu_v)$ (see 3.1.5) where $\nu_v = D_{\text{dR}}(\rho)$ and $\tau_p = 1$ is trivial.

Let $R_S$ be the universal type $S$ deformation ring on $\text{CNL}_G$, let $R^\square_S$ denote the universal type $S$ framed deformation ring, and let $R^\square_S = \widehat{\otimes}_{v \in S} \mathcal{R}_v$. Note that since $\pi$ has Iwahori fixed vectors for each $v | p$, the Galois type of $\rho|_{G_v}$ is trivial for each $v | p$ by local global compatibility (see part 2. of 2.2.1). After conjugating $\rho$, we may assume that $\rho$ takes values in $\text{GL}_2(O)$. Let $x : R_S[1/p] \rightarrow E$ denote the $E$-algebra morphism induced by $\rho$. By 1.4.4, we have $H^1_j(G_{F,S}, \text{ad}^0(\rho)) = 0$ if and only if $(R_S)^\wedge = E$. By part 1. of 2.2.5, the finite free $O$-module $S_{\kappa, \theta}(U)_m$ is naturally an $R_S$-module. By part 3. of 2.2.5, we conclude $(R_S)^\wedge = E$ if and only if $(S_{\kappa, \theta}(U)_m)^\wedge$ is a free $(R_S)^\wedge$-module. To show this, we will use 3.1.1.

Let $j = 4|S_f| - 1$. By part 2. of 1.4.3, the natural morphism $R_S \rightarrow \mathbb{R}_S$ is formally smooth of relative dimension $j$. We fix a presentation $\mathbb{R}_S \cong R_S[[z_1, \ldots, z_j]]$. We let $R = R^\square_S$ and let $\mathfrak{p}$ be the prime ideal of $R^\square_S$ that is the kernel of the composition of the surjection $R^\square_S \rightarrow R_S$ obtained by sending all $z_i$ to 0, with the surjection $R^\square_S[1/p] \rightarrow E$ induced by $\rho$. Setting $M = S_{\kappa, \theta}(U)_m \otimes_{R_S} R^\square_S$, we see that the $R$-module $M$ is finite free over $S = O[[z_1, \ldots, z_j]]$, and that $(S_{\kappa, \theta}(U)_m)^\wedge$ is a free $(R^\square_S)^\wedge$-module if and only if $M^\wedge$ is a free $R^\square_S$-module.

Set $h = \dim_E H^1(G_{F,S}, \text{ad}^0(\rho_E)(1))$, and $g = h - [F : Q] + |S_f| - 1$. Let $B = R^\square_S[[x_1, \ldots, x_g]]$. Let $B \rightarrow R$ be the morphism extending the canonical morphism $R^\square_S \rightarrow R$ by sending each $x_i$ to zero. Then, letting $\mathfrak{p}_B$ be the pullback of $\mathfrak{p}$ to $B$, the localization and completion of $B$ at $\mathfrak{p}_B$ is isomorphic to a power series ring over $(R^\square_S)^\wedge$ in $g$ variables. By part 2. of 2.2.5, $B_{\mathfrak{p}_B}$ is regular of dimension $g + 3|S_f| + [F : Q] = h + j$. We have shown that $B$, $R$, $M$, and $\mathfrak{p}$ satisfy the assumptions (a) through (d) of 3.1.1.

For every $n \geq 1$, let $Q_n$ be a set of places of $F$ as in 3.2.9. Note that $|Q_n| = h$ for all $n \geq 1$. We let consider the global deformation datum $S_{Q_n} = (S \cup Q_n, S_f, \mathfrak{p}, \theta^{-1}, (\mathcal{R}_v)_{v \in S_f})$ with $\mathcal{R}_v$ as above. We let $R_{S_{Q_n}}$ be the global type $S$ deformation ring, and let $R^\square_{S_{Q_n}}$ be the global type $S_{Q_n}$ framed deformation ring. There are natural surjections $R_{S_{Q_n}} \rightarrow R_S$ and $R^\square_{S_{Q_n}} \rightarrow R^\square_S$. The map $R_{S_{Q_n}} \rightarrow R^\square_{S_{Q_n}}$ is again formally smooth of dimension $j$. We set $R_n = R^\square_{S_{Q_n}}$. For each $v \in Q_n$, let $\Delta_v$ be the maximal $p$-power quotient of $\mathbb{F}_p^\times$, and let $\Delta_{Q_n} = \prod_{v \in Q_n} \Delta_v$. Fix a surjection $(\mathbb{Z}_p)^h \rightarrow \Delta_{Q_n}$. Letting $S_{\infty}$, $\mathfrak{a}$, and $\mathfrak{a}_n$ be as in the statement of 3.1.1, we have a surjection $S_{\infty} \rightarrow S[\Delta_{Q_n}]$. Then 1.4.10 and 1.4.11 show that $R_n$ is an $S_{\infty}$-algebra and that the kernel of the surjection $R_n \rightarrow R$ is generated by $\mathfrak{a}$. 

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We let $S_{\kappa,0}(U_{\Delta_{Q_n}})_{m_{Q_n}}$ be as in 2.1.2 and 2.1.3. Since we have assumed
\[(U(\mathbb{A}_{K}^\infty)^{\times} \cap l^{-1}D^X t)/F^X = 1\]
for any $t \in (D \otimes_F \mathbb{A}_{K}^\infty)^{\times}$, 2.1.3 implies that $S_{\kappa,0}(U_{\Delta_{Q_n}})_{m_{Q_n}}$ is a finite free $O[\Delta_{Q_n}]$-module and there is an $O$-module surjection $S_{\kappa,0}(U_{\Delta_{m}})_{m_{Q_n}} \to S_{\kappa,0}(U_{m})$ equivariant for all Hecke operators outside of $S \cup Q_n$, with kernel generated by $a$. By 2.2.7, $S_{\kappa,0}(U_{\Delta_{m}})_{m_{Q_n}}$ is an $R_{S_{Q_n}}$-module compatibly with its $O[\Delta_{Q_n}]$-module structure, and the $R_{S_{Q_n}}$-module structure on $S_{\kappa,0}(U_{m})$ via the surjection $S_{\kappa,0}(U_{\Delta_{m}})_{m_{Q_n}} \to S_{\kappa,0}(U_{m})$ is equal to the $R_{S_{Q_n}}$-module structure via the surjection $R_{S_{Q_n}} \to R_S$. Then we set $M_n = S_{\kappa,0}(U_{\Delta_{m}})_{m_{Q_n}} \otimes R_{S_{Q_n}} R_n$. Then $M_n$ is an $R_n$-module such that $M_n/a_n$ is free over $S_{\infty}/a_n$, and there is a surjection $M_n \to M$ with kernel $a_nR_n$, where the $R_n$-module structure on $M$ comes via the surjection $R_n \to R$.

It only remains to define a $B$-module structure on $R_n$, and verify parts (e), (f), and (g) of 3.1.1. For part (e), it suffices to show that the minimal number of generators for $R_n$ over $R_{S_{Q_n}} = R_{S_{Q_n}}^\text{loc}$ is bounded independently of $n$. By 1.4.6, it suffices to observe that the $F$-dimensions of the cohomology groups $H^1(G_{F,S\cup Q_n}, ad^0(\bar{\rho}))$ are bounded independently of $n$. Indeed, they are bounded above by
\[\dim_F H^1(G_{F,S}, ad^0(\bar{\rho})) + \sum_{\nu \in Q_n} \dim_F H^1(G_{\nu}, ad^0(\bar{\rho}))\]
and $\sum_{\nu \in Q_n} \dim_F H^1(G_{\nu}, ad^0(\bar{\rho}))$ is bounded above by $6h$.

Let $H^1_{S_{Q_n}}(G_{F,S\cup Q_n},{ad}_n^0)$ be as in §3.2. By 3.2.9, for every $n \geq 1$, there are maps of $O$-modules
\[\mathcal{O}/\varpi^n \to H^1_{S_{Q_n}}(G_{F,S\cup Q_n},{ad}_n^0)\]
whose kernels and cokernels are bounded independently of $n$, where
\[r = h - [F : Q] - \sum_{\nu \in S_F} \dim_E H^0(G_{\nu}, ad^0_{E}).\]

By 1.4.8, there are $O$-module maps
\[(\mathcal{O}/\varpi^n)^{\sigma} \to p_n/(p_{\nu}^{\sigma}, p_{\text{loc}}^\nu, \varpi^n)
\]
whose kernels and cokernels are bounded independently of $n$. Choose any $y_1, \ldots, y_{g} \in p_n$, whose image in $p_n/(p_{\nu}^{\sigma}, p_{\text{loc}}^\nu, \varpi^n)$ generate the image of the map (13). We define the morphism $B \to R_n$ by letting it be the canonical map on $R_{S_{Q_n}}^\text{loc} = R_{S_{Q_n}}^{loc}$ and sending $z_i$ to $y_i$. This $B$-algebra structure satisfies (f), and (g) of 3.1.1. We have now shown that all assumptions of 3.1.1 hold, so $M^\wedge_p$ is a finite free $R^\wedge_p$-module, which completes the proof the the theorem. \hfill $\square$

3.4. Smooth points on global deformation rings. We can now prove our second main theorem. Let $\overline{p} : G_{F,S} \to \text{GL}_2(F)$
be a continuous representation satisfying $\text{End}_{\mathcal{O}[G_{\varphi}]}(V_{\overline{p}}) = \mathbb{F}$, where $V_{\overline{p}}$ the representation space of $\overline{p}$. Let $\psi : G_{F,S} \to \mathcal{O}^{\times}$ be a continuous character such that $\psi = \det(\overline{p})$. Let $R_{\overline{p}}$ be the universal deformation ring for $\overline{p}$ on $\text{CNL}_O$, and let $R^\wedge_{\overline{p}}$ be the universal $\det = \psi$ deformation ring on $\text{CNL}_O$. Let $x$ be a closed point of $R^\wedge_{\overline{p}}[1/p]$, with residue field $k$ and let
\[\rho_x : G_{F,S} \to \text{GL}_2(k)\]
be a lift in the deformation class of the pushforward of the universal deformation via $x : R^\wedge_{\overline{p}}[1/p] \to k$. As in [Kis04], we can use the vanishing of $H^2_x(G_{F,S}, ad(\rho_x))$ to deduce smoothness of the universal deformation ring at $x$.

**Theorem 3.4.1.** Assume there is a finite totally real extension $F' / F$, a regular algebraic cuspidal automorphic representation $\pi$ of $\text{GL}_2(\mathbb{A}_F)$, and an isomorphism $\iota : \mathbb{C} \cong \mathbb{Q}_p$ such that the following hold.

1. $\rho_{\pi}|_{G_F} \cong \rho_{\pi,1}$,
2. $\overline{p}|_{G_{F'}}$ is absolutely irreducible.
3. If $\pi$ has complex multiplication by a quadratic extension $L/F'$, we assume $p > 2$ and $L \not\subseteq F'(\zeta_p)$.

Then the localization and completion $(R^\wedge_{\pi})_x$ is formally smooth over $E$ of dimension $1 + \delta_F + 2[F : Q]$, where $\delta_F$ denotes the Leopoldt defect for $F$ and $p$. If $\det(\rho_x) = \psi$, then the localization and completion $(R^\wedge_{\pi})_x$ is formally smooth over $E$ of dimension $2[F : Q]$. 

Proof. Since the Galois group of the maximal abelian pro-$p$ extension of $G_{F,S}$ has rank $1+\delta_F$, the result for $R_{\overline{\sigma}}$ follows from the result for $R_{\sigma}$ using 1.1.13. To prove the second statement, the proof of [Kis04, Theorem 8.2] carries over almost verbatim, replacing all the instances of $\mathbb{Q}$ with $F$. We give a slightly different, but closely related, proof below, in case it is of any interest.

For each $v|p$, let $\nu_v$ and $\tau_v$ be the $p$-adic Hodge type and Galois type, respectively, of $\rho_{x|G_v}$. Let $S$ be the global deformation datum

$$S = (S, \{v|p\}, \psi, (R^{\nabla}_v(\tau_v, \nu_v)(v|p)))$$

and let $R_S$ be the universal type $S$ deformation ring. Let $R^{\nabla}_{\overline{\sigma}, \psi}$ denote the universal $\{v|p\}$-framed deformation ring for deformations with determinant $\psi$. Let $R^{\nabla}_{loc} = \otimes_v R^{\nabla}_v(\tau_v, \nu_v)$ and $R^{\nabla}_{S} = \otimes_v R^{\nabla}_v(\tau_v, \nu_v)$. Fixing a lift in the deformation class of $\rho_{x}$, we again denote by $x$ the closed point induced by $\rho_{x}$ in the generic fibre of any of these rings. Let $a$ be the kernel of the surjection $(R^{\nabla}_{loc})_x^\wedge \to (R^{\nabla}_{S})_x^\wedge$. For each $v|p$, our assumptions imply that $\text{WD}(\rho_{x|G_v})$ is generic, by 1.2.3. Using [Kis09a, Lemma 3.4.12], we have that $(R^{\nabla}_{loc})_x^\wedge$ is formally smooth over $E$ of dimension $3[F: \mathbb{Q}] + 3\{v|p\}$, and $(R^{\nabla}_{S})_x^\wedge$ is formally smooth over $E$ of dimension $[F: \mathbb{Q}] + 3\{v|p\}$. This implies that the quotient must be given by modding out by a regular system of parameters of length $2[F: \mathbb{Q}]$, so the ideal $a$ is generated by $2[F: \mathbb{Q}]$ elements. By choosing a lift for the universal determinant $\psi$ deformation ring at each $v|p$, we get a local morphism $R^{\nabla}_{loc} \to R^{\nabla}_{S}$ such that $R_S = R^{\nabla}_{S} \otimes_{R^{\nabla}_{loc}} R^{\nabla}_{\overline{\sigma}, \psi}$. Then the surjection $(R^{\nabla}_{\overline{\sigma}, \psi})_x^\wedge \to (R_S)_x^\wedge$ is given by modding out by an ideal generated by $2[F: \mathbb{Q}]$ elements.

By 1.1.5 and part 4. of 1.1.9, we have

$$\dim(R^{\nabla}_{\overline{\sigma}, \psi})_x^\wedge \geq \dim_k H^1(G_{F,S}, \text{ad}^0(\rho_{x})) - \dim_k H^2(G_{F,S}, \text{ad}^0(\rho_{x})).$$

By absolute irreducibility, $H^0(G_{F,S}, \text{ad}^0(\rho_{x})) = 0$, so global Euler characteristic (see [Kis03, Lemma 9.7]) implies

$$\dim(R^{\nabla}_{\overline{\sigma}, \psi})_x^\wedge \geq 3[F: \mathbb{Q}] - \sum_{v|\infty} H^0(G_v, \text{ad}^0(\rho_{x})) = 2[F: \mathbb{Q}]$$

since $\det(\rho_{x}(c)) = -1$ for every choice of complex conjugation $c$.

By 3.3.1 and 1.4.4, $(R_S)_x^\wedge$ has trivial tangent space, so is a field. The lemma below applied to the surjection $(R^{\overline{\sigma}})_x^\wedge \to (R_S)_x^\wedge$, shows that $(R^{\nabla}_{\overline{\sigma}, \psi})_x^\wedge$ is formally smooth over $E$ of dimension $2[F: \mathbb{Q}]$. □

Lemma 3.4.2. Let $A \to B$ be a surjection of local Noetherian rings with $B$ regular. If the kernel of $A \to B$ is generated by elements $a_1, \ldots, a_r \in m_A$ with $r \leq \dim A - \dim B$, then $A$ is regular of dimension $\dim B + r$.

Proof. If $r = 1$, then $A/a_1 = B$ and the assumption $\dim A \geq \dim B + 1$ implies $\dim A/m_A m_A/m_A^2 \leq 1 + \dim A/m_A m_A/m_A^2 = 1 + \dim B \leq \dim A$, so $A$ is regular of dimension $\dim B + 1$. The general case proceeds by induction on $r$. □

Remark 3.4.3. The proof of 3.4.1 can be adapted to other quotients of the universal Galois deformation ring. Say we have a continuous absolutely irreducible representation

$$\rho : G_{F,S} \to \text{GL}_d(E)$$

which is potentially semistable at all $v|p$, and satisfies the conclusion of 3.3.1. Consider some prorepresentable deformation problem for $\rho$ on the category of local Artinian $E$-algebras with residue field $E$, and let $R$ be the prorepresenting object. If we know that $\dim R \geq d$, and that there are $d$ equations $f_1, \ldots, f_d \in m_R$ such that any deformation factorizing through $R/(f_1, \ldots, f_d)$ is deRham, then the argument in the proof of 3.4.1 using 3.4.2 and the vanishing of $H^1_g(G_{F,S}, \text{ad}(\rho))$ implies that $R$ is formally smooth over $E$ of dimension $d$.

One such case is that of ordinary (also called nearly ordinary) deformations. Another is the case of locally cyclotomic deformations considered by Hida (see [Hid09, §1]). One often knows the appropriate lower bound on the dimension by constructing a surjection from the deformation ring to one of Hida’s big ordinary (or nearly ordinary) Hecke algebras, and one can use the weight variables in the corresponding Iwasawa algebra for the $f_1, \ldots, f_d$. Indeed, if the weight variables are chosen so that modding out by them corresponds to the weight of our fixed potentially semistable Galois representation, then the resulting ring $R/(f_1, \ldots, f_d)$ parametrizes deformations of $\rho$ that for every $v|p$ are reducible with fixed inertial characters on the diagonal (in an upper triangular basis), with the Hodge–Tate weights “going the right way.”
such deformation is deRham, so the tangent space of $R/(f_1, \ldots, f_d)$ is a subspace of $H^1(G_{F,S}, \text{ad}(\rho)) = 0$, hence $R/(f_1, \ldots, f_d) = E$. Then 3.4.2 implies $R$ is formally smooth over $E$.

References


