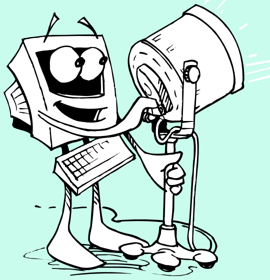

Spotlight on Coupled Springs: Normal Modes

Reference: Sections 6.3, 6.4 and WEB SPOTLIGHT ON MODELING: COUPLED SPRINGS

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In Sections 6.3 and 6.4 we used the eigenvalues and eigenvectors of a matrix to construct solutions of a first-order, undriven linear system. Now we use this eigen-element approach to find the solutions of the *second-order linear system*

$$z'' = Az \tag{1}$$

where z is a column n-vector and A is an $n \times n$ matrix of real constants.

THEOREM 1

General Solution of $z'' = Az$

Suppose that the constant $n \times n$ matrix A with real entries has n distinct and nonzero eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v^1, \ldots, v^n$. Then the general solution of the second-order system $z'' = Az$ is

$$z = v^1(c_1e^{r_1t} + c_2e^{-r_1t}) + \cdots + v^n(c_{2n-1}e^{r_nt} + c_{2n}e^{-r_nt}) \tag{2}$$

where $c_1, c_2, \ldots, c_{2n}$ are arbitrary constants and $r_1 = \sqrt{\lambda_1}, \ldots, r_n = \sqrt{\lambda_n}$.

☞ If any $\lambda_k < 0$, then the r_k and the c_j are complex numbers.

Proof. We outline the idea of the proof, but omit the details. The function vector $z = ve^{rt}$ is a solution of $z'' = Az$ if $r^2 = \lambda$ is a nonzero eigenvalue of A and v is a corresponding eigenvector because

$$z'' = r^2ve^{rt}$$

$$Az = Ave^{rt} = r^2ve^{rt}$$

The same argument shows that $z = ve^{-rt}$ is also a solution, so we have a pair of independent solutions $\{ve^{rt}, ve^{-rt}\}$ for each eigenvalue λ and corresponding eigenvector v. The basic solution set $\{v^1e^{r_1t}, v^1e^{-r_1t}, \ldots, v^ne^{-r_nt}\}$ has $2n$ elements and $z'' = Az$ has the $2n$ state variables $z_1, z_1', \ldots, z_n, z_n'$. ■

☞ We deal with $z'' = Az$ directly, rather than with the equivalent first-order system $z' = y$, $y' = Az$.

Now let's apply all this to the second-order system introduced in the WEB SPOTLIGHT ON MODELING: COUPLED SPRINGS to model the motion of a coupled spring-block system. In the process, we are led to some interesting two-dimensional orbital portraits.

EXAMPLE 1

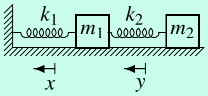
Coupled Springs and Normal Modes

The second-order system

$$\begin{aligned} x'' &= -(a+b)x + by \\ y'' &= cx - cy \end{aligned} \tag{3}$$

models the vibrations of the coupled spring-block system sketched in the margin. The positive constants a, b, and c are

$$a = k_1/m_1, \qquad b = k_2/m_1, \qquad c = k_2/m_2 \tag{4}$$



☞ See the WEB SPOTLIGHT ON MODELING: COUPLED SPRINGS

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Write (3) as $z'' = Az$, where $z = [x \quad y]^T$ and the matrix A is $\begin{bmatrix} -a-b & b \\ c & -c \end{bmatrix}$. The eigenvalues of A are the roots λ_1, λ_2 of the characteristic polynomial

$$p(\lambda) = \lambda^2 + (a+b+c)\lambda + ac$$

$$\lambda_1, \lambda_2 = -\frac{1}{2}(a+b+c) \pm \frac{1}{2}\left((a+b+c)^2 - 4ac\right)^{1/2}$$

☞ The key to showing this inequality is that $(a-c)^2 \geq 0$ for all a and c.

Since a, b and c are positive and $(a+b+c)^2 > 4ac$, both eigenvalues λ_1 and λ_2 are negative, and their square roots are the pure imaginary numbers $\pm i\omega_1$, $\pm i\omega_2$ where $\omega_1 = \sqrt{-\lambda_1}$ and $\omega_2 = \sqrt{-\lambda_2}$. In the notation of Theorem 1 with v^1 and v^2 the eigenvectors of A corresponding to λ_1 and λ_2, the general complex-valued solution of the second-order system (3) is

$$z = \begin{bmatrix} x \\ y \end{bmatrix} = v^1\left(c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}\right) + v^2\left(c_3 e^{i\omega_2 t} + c_4 e^{-i\omega_2 t}\right) \tag{5}$$

where c_1, c_2, c_3, and c_4 are arbitrary complex constants. Extract the general real-valued solution from (5):

$$z = v^1(C_1 \cos\omega_1 t + C_2 \sin\omega_1 t) + v^2(C_3 \cos\omega_2 t + C_4 \sin\omega_2 t) \tag{6}$$

☞ Example 3.4.1 shows how to derive this form of equation (6).

where C_1, C_2, C_3, C_4 are arbitrary real constants. Furthermore, the general solution (6) has the amplitude and phase-angle form

$$z = K_1 v^1 \cos(\omega_1 t + \theta_1) + K_2 v^2 \cos(\omega_2 t + \theta_2) \tag{7}$$

where K_1, K_2, θ_1, θ_2 are arbitrary real constants.

The eigenvectors v^1 and v^2 in the xy-plane are *normal mode vectors* and ω_1 and ω_2 are the respective *normal mode circular frequencies*. The general solution (7) is an arbitrary linear combination of the *normal mode oscillations* $v^1 \cos(\omega_1 t + \theta_1)$ and $v^2 \cos(\omega_2 t + \theta_2)$.

Let's put numbers into the picture and plot specific normal mode oscillations and some of their linear combinations.

EXAMPLE 2

Portraits of the Motions of Coupled Springs

In system (3) take these values for the masses and the spring constants:

$$m_1 = 4 \text{ kg}, \quad m_2 = 1 \text{ kg}, \quad k_1 = 40 \text{ kg/sec}^2, \quad k_2 = (40/3) \text{ kg/sec}^2$$

Then the ratios in (4) are

$$a = k_1/m_1 = 10, \quad b = k_2/m_1 = 10/3, \quad c = k_2/m_2 = 40/3$$

System (3) becomes

$$\begin{bmatrix} x \\ y \end{bmatrix}'' = A\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where } A = \begin{bmatrix} -40/3 & 10/3 \\ 40/3 & -40/3 \end{bmatrix}$$

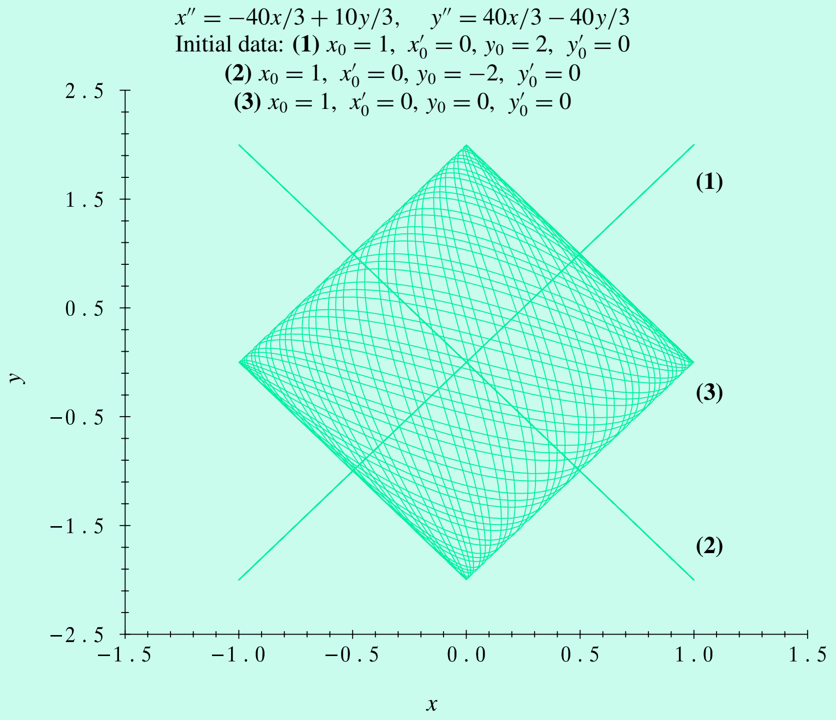

FIGURE 1 Normal mode oscillations and a Lissajous curve (Example 2).

The characteristic polynomial of A is $p(\lambda) = \lambda^2 + 80\lambda/3 + 400/3$. The eigenvalues and normal mode circular frequencies are

$$\lambda_1 = -20/3, \quad \lambda_2 = -20, \quad \omega_1 = \sqrt{20/3}, \quad \omega_2 = \sqrt{20}$$

The normal modes v^1 and v^2, two normal mode oscillations, and the general solution z are, respectively,

$$v^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}; \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(\sqrt{20/3}t), \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cos(\sqrt{20}t)$$

$$z = \begin{bmatrix} x \\ y \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(\sqrt{20/3}t + \theta_1) + K_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cos(\sqrt{20}t + \theta_2)$$

Figure 1 shows two normal mode oscillations: **(1)** $K_1 = 1,\ \theta_1 = 0,\ K_2 = 0$, and **(2)** $K_1 = 0,\ K_2 = 1,\ \theta_2 = 0$. The figure also shows a linear combination of normal mode oscillations: **(3)** $K_1 = K_2 = 1/2,\ \theta_1 = \theta_2 = 0$. The latter is a *Lissajous curve*. The corresponding oscillation is *not* periodic because the ratio of the two normal mode frequencies is $\omega_1/\omega_2 = 1/\sqrt{3}$, which is irrational (see Section 3.4). The Lissajous curve never returns to its starting point as it wanders through a rectangle whose sides are parallel to the normal modes.

☞ The Lissajous curve in Figure 1 is that of Figure 4 in the WEB SPOTLIGHT ON MODELING: COUPLED SPRINGS , but extended over a much longer span of time!

We will interpret the curves and lines in Figure 1 as a display of how the positions x and y of the two blocks on the springs play off against each other, given different sets of initial data. The oscillations along the normal modes reflect the fact that if the initial positions of the blocks are in the ratio $1:2$ or $1:(-2)$ and the initial velocities are both zero, then the blocks oscillate periodically and their positions always maintain the initial ratio. But if the initial ratio of the positions is anything other than $1:2$ or $1:(-2)$, a wandering Lissajous curve must result.

The curves in Figure 1 intersect each other, but this does not violate the property that distinct orbits of an autonomous system can't meet because the $xx'yy'$ state space of the orbits is four-dimensional. In Figure 1 we see the projection of three non-intersecting orbits from this four-dimensional state space into the two-dimensional space of the xy-plane where the projections *do* intersect.

PROBLEMS

1. *Coupled Springs* Suppose that $m_1 = 1,\ k_1 = k_2 = 1$ in (4), but the second mass m_2 may take on various values. Calculate the normal modes and frequencies for $m_2 = 1/2,\ 1,\ 3/2$. In each case plot a pair of normal mode oscillations and a Lissajous curve to obtain a figure resembling Figure 1.

2. *Laplace Transforms of Coupled-Springs IVPs* Use the Laplace transform (Chapter 5) to solve the system of linear second-order ODEs $x'' = -40x/3 + 10y/3,\ \ y'' = 40x/3 - 40y/3$, where $x(0) = 1,\ x'(0) = 0,\ y(0) = 2,\ y'(0) = 0$. Repeat with $x(0) = 1,\ x'(0) = 0,\ y(0) = 0,\ y'(0) = 0$. Compare with the results in Examples 1 and 2.