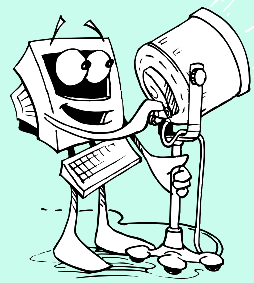
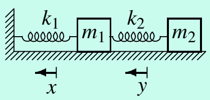
Spotlight on Modeling: Coupled Springs



Reference: Sections 3.1 and 6.4.

In Chapter 3 we saw how to model the motion of a block attached to a single spring under assumptions about the spring force and other forces acting on the block. That model involves a single second-order ODE that is equivalent to a normal first-order system with the position and velocity of the block as state variables. The motions of multiple coupled springs and blocks also lead to a modeling system of normal first-order ODEs.



Suppose that a coupled system of two blocks and two springs is attached to a wall, and the blocks slide back and forth on a smooth horizontal table (see the margin figure). At equilibrium, the springs are neither stretched nor compressed. Let's measure the respective displacements of the blocks from their equilibrium positions by x and y, with the positive direction indicated by the arrows.

Suppose that air resistance and sliding friction are negligible, so the only forces acting on each block are gravity, the upward force of the table, and the spring forces. Gravity and the upward force of the table are equal and opposite (Newton's Third Law, Section 4.1), so we can ignore them. Suppose that each spring force is proportional to the spring's displacement from its equilibrium length (Hooke's Law) and acts in a direction to restore the spring to its equilibrium length.

In particular, if the displacements of the blocks are x and y, then the spring force acting on the block of mass m_2 is $-k_2(y-x)$, where k_2 is the positive Hooke's Law constant for the spring. To check that the algebraic sign of the spring force is correct, observe that if $y > x$ then the second spring is compressed and the spring force acts to decompress the spring. In this case the spring force on m_2 should be directed to the right and should be negative (which it is). On the other hand, the spring connecting the two blocks exerts a force $k_2(y-x)$ on m_1 that is equal and opposite in sign to the force it exerts on m_2 (Newton's Third Law). Hooke's Law implies that m_1 is also subjected to the spring force $-k_1 x$.

☞ Use free-body diagrams to derive these ODEs.

Apply Newton's Second Law (Section 4.1) to each block:

$$\begin{aligned} m_1 x'' &= -k_1 x + k_2(y-x) = -(k_1+k_2)x + k_2 y \\ m_2 y'' &= -k_2(y-x) = k_2 x - k_2 y \end{aligned} \tag{1}$$

System (1) models the vibration dynamics of the coupled spring-block configuration.

Divide through by the masses to put system (1) in *normal form*. To make this normalized *second-order* system acceptable to numerical solvers convert it to an equivalent normal *first-order* system.

EXAMPLE 1

The First-Order System that Models Coupled Springs/Blocks

The state variables for the coupled spring-block configuration modeled by system (1) are the position and velocity of each of the two blocks. Introduce new names x_1, x_2, x_3, and x_4 for the four state variables:

$$x_1 = x, \qquad x_2 = x', \qquad x_3 = y, \qquad x_4 = y'$$

Then system (1) is equivalent to the undriven first-order linear system

$$\begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = -\left(\frac{k_1+k_2}{m_1}\right)x_1 + \frac{k_2}{m_1}x_3 \\ x_3' &= y' = x_4 \\ x_4' &= y'' = \frac{k_2}{m_2}x_1 - \frac{k_2}{m_2}x_3 \end{aligned} \qquad (2)$$

System (2) is in normal form.

EXAMPLE 2

Linear System Notation for Coupled Springs and Blocks

Take system (2) for a pair of coupled springs and blocks and specify some initial conditions. The corresponding IVP, $x' = Ax$, $x(0) = x^0$, is

$$x' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1+k_2}{m_1} & 0 & \dfrac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{m_2} & 0 & -\dfrac{k_2}{m_2} & 0 \end{bmatrix} x, \qquad x(0) = x^0$$

The system matrix A is the 4×4 constant matrix of coefficients given above.

Geometry of Solutions

Sometimes it is difficult to find solution formulas for differential systems that model intricate natural processes. This is why the focus in this section is on qualitative properties of solutions.

First, we review some notions about solutions of a system $x' = f(t, x)$, where x and f are n-vectors. A function vector $x(t)$ defined on some t-interval I where t_0 is a *solution* if $x'(t) = f(t, x(t))$ for all t in I. A solution of the system determines curves whose behavior highlights properties of the solution.

❖ **Curves Associated with Solutions.** Suppose that $x = x(t)$ is a solution of the system $x' = f(t, x)$. The point $(t, x(t))$ traces out a *time-state curve* in the *time-state space* $\mathbb{R}^{n+1}$ of the variables $t, x_1, x_2, \ldots, x_n$. The projection of a time-state curve onto the tx_j-plane is the x_j- *component curve*. The projection of a time-state curve onto the $x_1, x_2, \cdots, x_n$ *state space* is an *orbit*. A collection of orbits is an *orbital portrait*.

Let's illustrate these concepts with graphs associated with the spring-block system.

EXAMPLE 3

☞ See the library entries for *Two Linear Springs* under Physical Models.

ODEA Oscillating Springs

What happens if you compress the springs of Example 1 by pushing the first block one

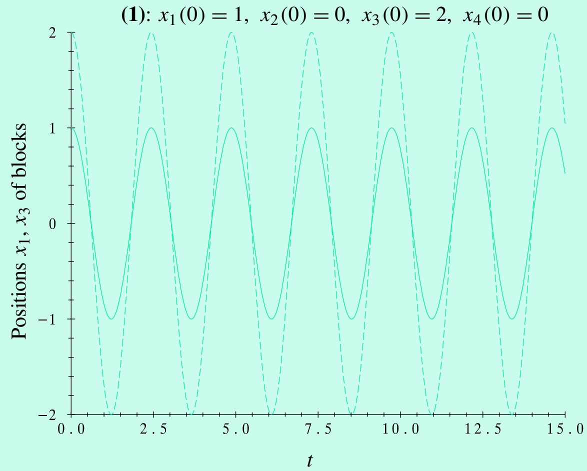


FIGURE 1 In-phase periodic oscillations of the coupled spring-block system (Example 3): first block (solid), second block (dashed).

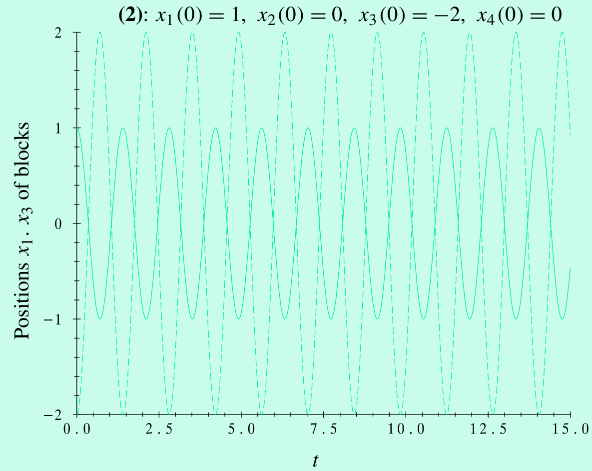


FIGURE 2 Out-of-phase periodic oscillations of the coupled spring-block system (Example 3): first block (solid), second block (dashed).

meter to the left ($x_1(0) = 1$) and the second block two meters to the left ($x_3(0) = 2$) and then release the blocks from rest ($x_2(0) = 0$ and $x_4(0) = 0$)? Let's suppose that

$$m_1 = 4 \text{ kg}, \quad m_2 = 1 \text{ kg}, \quad k_1 = 40 \text{ kg/sec}^2, \quad k_2 = \frac{40}{3} \text{ kg/sec}^2$$

From Example 2 the modeling IVP with these initial conditions is

$$\begin{aligned} x_1' &= x_2, & x_1(0) &= 1 \\ x_2' &= -\frac{40}{3}x_1 + \frac{10}{3}x_3, & x_2(0) &= 0 \\ x_3' &= x_4, & x_3(0) &= 2 \\ x_4' &= \frac{40}{3}x_1 - \frac{40}{3}x_3, & x_4(0) &= 0 \end{aligned} \tag{3}$$

The tx_1- and tx_3-component curves of IVP (3) over a 15-sec time interval appear in Figure 1. The springs oscillate periodically (and in phase) about their equilibrium.

Repeat the process, but push the first block one meter to the left and the second block two meters to the right of equilibrium (so $x_1(0) = 1$ and $x_3(0) = -2$) and release the blocks from rest. Now the periodic oscillations are out of phase and the frequency is higher (Figure 2). Finally, move the first block one meter to the left ($x_1(0) = 1$) but keep the second block in its equilibrium position ($x_3(0) = 0$) and release the blocks from rest. Figure 3 shows the nonperiodic oscillations of each of the two masses.

Figure 4 shows the positions of the two masses relative to one another in all three cases. The straight line segments **(1)** and **(2)** correspond to the two periodic oscillations of Figures 1 and 2; they are the *normal modes*. See the WEB SPOTLIGHT ON COUPLED SPRINGS: NORMAL MODES for more on normal modes. The motion of the two springs in Figure 3 shows up in Figure 4 as the *Lissajous curve* **(3)** that wanders through a

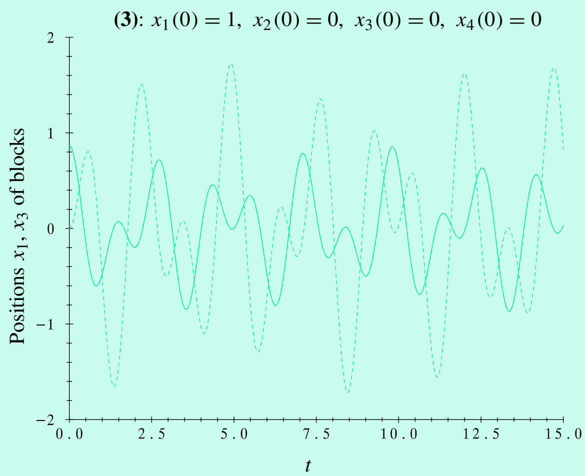


FIGURE 3 Both masses oscillate, but not periodically (Example 3): first block (solid), second block (dashed).

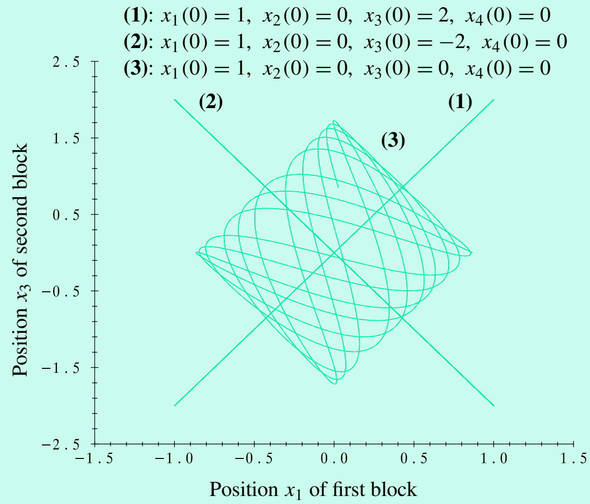


FIGURE 4 Periodic in-phase oscillations along line **(1)**, out of phase along **(2)**; nonperiodic along the Lissajous curve **(3)** (Example 3).

parallelogram.[12]

See the WEB SPOTLIGHT ON COUPLED SPRINGS: NORMAL MODES for actual solution formulas. The spotlight also reveals why we used the initial data $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = \pm 2, 0$, $x_4(0) = 0$.

We cannot show graphs of orbits or time-state curves for the model system of Example 3 because we would need the four-dimensional $x_1x_2x_3x_4$-space for the former at the five-dimensional $tx_1x_2x_3x_4$-space for the latter. However, the curves in Figure 1–4 are projections of these curves in four and five dimensional spaces into various planar spaces where we can see what happens.

The coupled spring system is linear, undriven and has constant coefficients, so the methods of Section 6.4 apply. In particular, the eigenvalues of the matrix of coefficients of the system in IVP (3) are $\pm i\sqrt{20}$ and $\pm i\sqrt{20/3}$, and the solutions involve sinusoids of natural frequencies $\sqrt{20}$ and $\sqrt{20/3}$ (Problem 12). The graphs in Figures 1–4 strongly suggest sinusoidal solutions. The graphs also suggest that the amplitudes do not decay; that is because the system has no damping terms.

PROBLEMS

Hooke's Law Spring (Review). The system $x' = y$, $y' = -bx - ay + A\cos\omega t$, $a \geq 0$, $b > 0$, is equivalent to the scalar ODE $x'' + ax' + bx = A\cos\omega t$ that models the motion of a block attached to a damped and driven Hooke's Law spring. Find the solution $x(t)$, $y(t)$ of each IVP by using the techniques of Chapter 3 to solve the equivalent scalar IVP.

[12] The 19th century French applied mathematician Jules Lissajous visualized the curves by reflecting light from mirrors on vibrating tuning forks onto a screen.

1. $x' = y, \quad y' = -x - y; \quad x(0) = 1, \ y(0) = 0.$

2. $x' = y, \quad y' = -x - 3y; \quad x(0) = 1, \ y(0) = 0.$

3. $x' = y, \quad y' = -25x + \cos 5t; \quad x(0) = 0, \ y(0) = 0.$

4. $x' = y, \quad y' = -25x + \cos(5.5t); \quad x(0) = 0, \ y(0) = 0.$

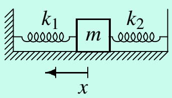
More Portraits of Orbits. Plot x- and y-component graphs, the orbit, and the time-state curve. Interpret what you see in terms of the behavior of the spring as t increases.

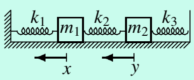
5. Problem 1, $|x| \le 1$, $-0.75 \le y \le 0.25$, $0 \le t \le 15$.

6. Problem 2, $0 \le x \le 1$, $-0.3 \le y \le 0$, $0 \le t \le 10$.

7. Problem 3, $|x| \le 1$, $|y| \le 5$, $0 \le t \le 10$. **8.** Problem 4, $|x| \le 2$, $|y| \le 2$, $0 \le t \le 30$.

Multiple Springs.





9. *One Block and Two Springs* Explain why the linear system, $x' = y$, $y' = -(k_1 + k_2)(x/m)$ models the undamped motion of the block-and-springs arrangement shown in the margin figure. Then create a model with damping whose magnitude is proportional to velocity.

10. *Two Blocks and Three Springs (with Damping)* Assume Hooke's Law and damping proportional to velocity; write a model system of two, second-order, linear, undriven ODEs for the system of springs and blocks shown in the margin figure. Then create an equivalent first-order system. [*Hint*: look at Problem 9.]

11. *Two Blocks and Three Springs* Set $m_1 = 4$ kg, $m_2 = 1$ kg, $k_1 = 40$ kg/sec^2, $k_2 = 40/3$ kg/sec^2, $k_3 = 20/3$ kg/sec^2 in the system that models the system of springs and blocks shown in the margin for Problem 10. Assume no damping and experiment with initial data and create pictures that resemble those shown in Figures 1–4. Then repeat, but with damping coefficients $c_1 = c_2 = 1$ kg/sec. Interpret each graph in terms of the motions of the blocks.

12. *Solution Formulas for Spring Systems* Find the solution formula for the IVP (3). [*Hint*: use the Method of Eigenvectors in Section 6.4 to solve the system.]

Sensitivity to Changes in the Data.

13. *Coupled Hooke's Law Springs: Response to Parameter Changes* In system (2) take the parameter values $(k_1 + k_2)/m_1 = 1$, $k_2/m_1 = \alpha$, $k_2/m_2 = 1$, where the magnitude of the positive constant α is m_2/m_1. For these values, system (2) becomes

$$x_1' = x_2, \ x_2' = -x_1 + \alpha x_3, \ x_3' = x_4, \ x_4' = x_1 - x_3$$

For each value of $\alpha = 0.05, 0.5, 0.95$ plot

- tx_1- and tx_3-component graphs for initial data $(\sqrt{\alpha}, 0, 1, 0)$, and $(-\sqrt{\alpha}, 0, 1, 0)$.
- The projections onto $x_1x_3x_2$-space of the orbit with initial data $(2\sqrt{\alpha}, 0, 0, 0)$.
- The x_1x_3 Lissajous graphs for initial data $(\pm\sqrt{\alpha}, 0, 1, 0)$ and $(2\sqrt{\alpha}, 0, 0, 0)$.

Use $0 \le t \le 25$ and describe for each value of α how the spring system behaves for each of the three sets of initial values. What happens to the frequency of the oscillations as α increases from 0.05 to 0.95? What happens to the amplitude?