

## Test 1

Each question is worth 20 points. Please show all your work.

1. Consider the initial value problem

$$(t - 2)y' + y = 1; \quad y(0) = y_0.$$

- a.) Find the solution.  
b.) There is only one value for  $y_0$  so that the solution does not escape in finite time. What is it?

2. A cup of coffee at 90 degrees  $C$  is placed into an oven at 200 degrees. The oven is then shut off and begins to cool. It is in a room kept at 20 degrees. Assume the cooling (or warming) is subject to Newton's law.

- a.) Write a system of differential equations for the temperature of the oven and the coffee. Your answer should involve two separate constants of proportionality. Be sure to include the initial conditions.  
b.) Are your proportionality constants positive or negative? Explain.  
c.) Argue from your system that as  $t \rightarrow \infty$  both oven and coffee will approach 20 degrees.

3. Consider the second order ODE

$$y'' + 4y' + 4y = 0.$$

Suppose  $y(0) = 1$  and  $y(a) = 0$  for some  $a > 0$ . Show  $y'(0) < 0$ .

4. In this problem we will study a population subject to the logistic equation

$$y' = y(6 - y).$$

- a.) Draw the phase portrait, indicating the equilibrium solutions and where the population will be rising or falling.  
b.) Suppose  $y(0) = 3$ . Use the phase portrait to predict the value of  $y$  at  $t \rightarrow \infty$ .  
c.) Solve the differential equation subject to  $y(0) = 3$ .

## Answers

1.a)  $y = \frac{t - 2y_0}{t - 2}$

b.) When  $y_0 = 1$ ,  $\lim_{t \rightarrow 2} y = 1$ . If  $y_0 \neq 1$ , then  $\lim_{t \rightarrow 2} y = \pm\infty$ .

2.a.) Let  $x$  and  $y$  be the temperature of oven and coffee respectively. We have the two equations

$$\begin{aligned}x' &= k_1(x - 20) \\y' &= k_2(y - x)\end{aligned}$$

We have  $k_1 < 0$  since  $x > 20$  and the oven is cooling. We have  $k_2 < 0$  as the coffee is cooler than the oven and warming, at least initially. Finally, we have  $x(0) = 200$  and  $y(0) = 90$ . Note: there are other ways to write these equations so that the signs  $k_1$  and  $k_2$  are different.

b.) At steady state  $x' = 0 = y'$  so  $x = y$  (from the second equation) and  $x = 20$  (from the first).

3. We have

$$y(t) = c_1 t e^{-2t} + c_2 e^{-2t}.$$

Also  $c_2 = y(0) = 1$ . Then

$$0 = y(a) = c_1 a e^{-2a} + e^{-2a}$$

so  $c_1 = -1/a$ . Then

$$y' = (-1/a)e^{-2t} + (2/a)te^{-2t} - 2e^{-2t}$$

so  $y'(0) = (-1/a) - 2 < 0$ .

4.a) The equilibrium solutions are  $y = 0, 6$ . The population is rising if  $0 < y < 6$  and falling if  $y > 6$ .

b)  $y \rightarrow 6$ .

c.)  $y = 6e^{6t}/(1 + e^{6t}) = 6/(1 + e^{-6t})$ .