Each question is worth 20 points.

1. A population of fish grows at a rate of 2 percent a year. Let \( y = y(t) \) be the population of fish at year \( t \) measured in millions. At year \( t = 0 \) there are 200 million fish, but at year \( t = 20 \) constant rate harvesting begins at a rate of 10 million fish per year.
   a.) Explain why the IVP
   \[
   y' - 0.02y = -10 \text{ step}(t - 20), \quad y(0) = 200
   \]
   models this scenario.
   b.) Solve for \( y(t) \).

2. Consider the undamped spring system subjected to two shocks, as indicated in the following ODE:
   \[
   y'' + 4y = A\delta(t - \pi) - \delta(t - 2\pi).
   \]
   The constant \( A \) is to be determined. Suppose \( y(0) = 0 \) and \( y'(0) = 0 \).
   a.) Find the solution \( y(t) \), \( t > 0 \).
   b.) Is there a value of \( A \) that guarantees \( y(t) = 0 \) for \( t \geq 2\pi \)? Explain.

3. Solve the partial differential equation
   \[
   x \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x^2
   \]
   subject to the initial condition \( u(x, 0) = 0 \) and the boundary condition \( u(0, t) = 0 \). You might want to use the Laplace transform.

4. A hollow cylinder of stainless steel has internal radius 1cm and outer radius of 10 cm. The temperature of the inside is 100 degrees C and the outside is 20 degree C. The top and bottom are insulated, so heat passes radially from the inside to the outside. Let \( T = T(x) \) be the temperature at distance \( x \) from the center. The surface area of a cylinder is \( 2\pi rh \).
   a.) Give an argument to show \( \frac{d}{dx} (x \frac{dT}{dx}) = 0 \).
   b.) Solve for \( T(x) \).
Answers

1.b.) $y(t) = 200e^{0.02t} + 500 \text{step}(t - 20)(1 - e^{0.02(t-20)})$.

2. $y = \frac{A}{2} \text{step}(t - \pi) \sin(t - \pi) - \frac{1}{2} \text{step}(t - 2\pi) \sin(t - 2\pi)$, so $A = 1$ will make $y(t) = 0$ for $A \geq 2\pi$.

3. $u(x, t) = \frac{x^2}{2} (1 - e^{-2t})$. After doing the Laplace transform, get an equation

$$v_x + (s/x)v = x$$

which has integrating factor $x^s$. Then

$$v = \frac{x^2}{s(s + 2)} = \frac{x^2}{2} \left( \frac{1}{s} - \frac{1}{s + 2} \right).$$

4.b.) $T(x) = -\frac{80}{\ln(10)} \ln(x) + 100$. 