In this project we will examine the differential equation

\[ ay'' + by' + cy = F(t) \]  

with all the constants \(a\), \(b\), and \(c\) positive, or at least non-negative. We will always take \(F(t)\) to be bounded and periodic – or perhaps a sum of periodic functions. The main question we will address is the following:

**Problem:** How are the forcing function \(F(t)\) and the steady state solution related?

We will regard \(F(t)\) as the input and the steady state solution as the output. Suppose we know the input, can we say anything about the output? Conversely, suppose we can measure the output, can we say anything about the input?

We developed equations such as (1) to model spring systems; they appear equally in modeling simple electric circuits. See question 9 below. There the question is to compare input and output frequencies and amplitudes.

The material for this project comes from Borelli and Coleman, §4.3 and §4.4, and two “spotlights”, one starting on p. 137 and the other on p. 293. Remember, your write-up should be a narrative: tell me a story. More technically, you should graph inputs and outputs so you can compare them.

1. (Matlab) In this problem we show how to how to isolate the steady state solutions. Use Matlab to graph the solutions of

\[ y'' + \frac{1}{2}y' + \frac{257}{16} y = 100 \cos(t) \]

with \(y'(0) = 0\) and \(y(0) = 0\), then \(y(0) = 12\) for \(0 \leq t \leq 50\). (See p. 274). Notice that if we graphed either of these for \(25 \leq t \leq 50\), we would get the same graph, for all the transients have died out. (You can adjust the axes using Matlab’s graphing tool.) Follow up this observation by graphing the steady state solution of

\[ y'' + \frac{1}{2}y' + \frac{257}{16} y = 100 \text{square}(t). \]

Note that the output is not a square wave; indeed, what sort of function is it? You may have to run for longer time to eliminate the transients.

Recall that the function \(\text{square}(t)\) is periodic of \(2\pi\) and

\[ \text{square}(t) = \begin{cases} 
+1, & 0 \leq t < \pi; \\
-1, & \pi \leq t < 2\pi.
\end{cases} \]
It’s built into Matlab.

2. (Matlab) If the input function is periodic, then so is the output. However, the amplitude can change and the function can be shifted. (These phenomena are known respectively as gain and phase shift; see p. 276.) Do problem 9 on p. 280 to see this in action.

3. Show that the steady state solution of

\[ ay'' + by' + cy = F_0 \cos(\omega t) \]

is of the form

\[ F_0 \left( \frac{c - a\omega^2}{(c - a\omega^2)^2 + b^2\omega^2} \right)^{1/2} \cos(\omega t + \phi) \].

(See pp. 275-76 and p. 288, where similar problems are solved.) Suppose \( a = 1 \). Suggest strategies for increasing the amplitude of the output function. This is “tuning” the system.

4. (Matlab) Consider

\[ y'' + by' + cy = 10 \cos((1/2)t) \]

Examine what happens as \( b \) gets small and \( c \) gets near \((1/2)^2 = 1/4\). Choose various values of \( b \) and \( c \) to illustrate your conclusions. You’ll have to choose initial conditions (such as \( y(0) = 0 = y'(0) \)) to get solutions, but graph for large \( t \) so that all the transients have died out.

5. (Matlab) Now see if this all works equally well for non-continuous periodic functions. Specifically, consider

\[ y'' + by' + cy = 10 \text{square}((1/2)t) \]

Use the same values of \( b \), \( c \), and \( t \) as in the previous problem.

The next few problems concern the question of tuning an electric system. The current \( I = I(t) \) in a simple electric circuit can be modeled by the equation

\[ LI'' + RI' + \frac{1}{C} I = E'(t) \]  \hspace{1cm} (2)

with \( L \), \( R \), and \( C \) positive constants and \( E(t) \) is the input voltage. (See §4.4.) Here the question is to tune the circuit by varying \( C \) (or \( R \)) to isolate input frequencies. The constants \( R \) and \( C \) are associated to a resistor and a capacitor respectively, both easily adjusted parts of the circuit. If you are doing question 9 below, you might want to insert the answer here as part of your narrative.

6. (Matlab, see p. 293). We now take an input function with multiple frequencies and try to tune the system to isolate one of them. Consider the differential equation

\[ y'' + 0.1y' + \frac{1}{C} y = \sin(t) + 4 \sin(5t) \].
Try the values $C = 1$, $1/25$, and $C = 1/81$ and graph the output functions $y$ for large $t$ – after the transients have died out. Which value of $C$ isolates which frequency? Use your answer to no. 3 to explain why this works.

7. (Matlab, adapted from p. 294, no. 1.) Repeat no. 6, but now varying the resistance. Take

$$y'' + Ry' + 25y = \sin(t) + 4\sin(5t)$$

and try $R = 1.0$, $0.1$, and $0.01$. Which frequency do you pick up? Again explain your answers using no. 3 to explain what’s going on. Is it better to tune $R$ or $C$?

8. (Matlab, adapted from p. 294, no. 2.) This is about interference. Take

$$y'' + Ry' + \frac{1}{C} y = 4\sin(t) + \sin(\omega t).$$

Try to isolate the $\sin(t)$ input term in the output by tuning the circuit to that frequency. Take $R = 0.1$ and $1/C = 1$. What happens if we take $\omega = 1.01$ and $1.1$; that is, values of $\omega$ near $1$? (You should see a “beat” phenomenon; this is the interference.) How large to do you need to take $\omega$ to make this negligible – that is, hard to see on the graph?

9. (Optional and extra credit) Develop the mathematical model for a simple circuit; that is, derive Equation (2) above. The simple circuit is pictured on p. 283; it has a resistor, a conductor, an inductor, and an external voltage. The necessary principles of electricity are presented on pp. 282-5 and the derivation of the equation is on p.p. 285-6. You will first have to develop the equation for the charge across the capacitor. See Equation (3) on p. 286. Since this is extra credit, I want to see more than a simple recapitulation of the material in the book: add your own insights.