MATH 360-1, Fall 2008 **MENU Applied Analysis**

Project 3

This project is adapted from Chapter 8.7 of our text and from Chapter 5.5 of

Borelli, R.L. and Coleman, C.S, *Differential Equations: A Modeling Perspective*, John Wiley and Sons, Inc., New York 1998.

In Project 2 we modeled the course of a disease through a population. In that case, we assumed the disease was relatively benign – all ill members of the population recover. Here we will model a different, far more insideous type of disease: one which kills, but slowly, so that ill members of the populations can move about an infect healthy members of the population. Human examples include tuberculosis; the text uses tuberculosis in New Zealand possums for data.

Assumptions: We have a contagious disease which is eventually fatal; that is, ill members of the population never recover. We will divde the population into two parts:

$$S = \textit{Susceptibles}\text{: those can catch the disease}$$
$$C = \textit{Contagious}\text{: the sick who can transmit the disease}$$

Thus $P = S + C$ is the entire population. Here are the basic assumptions:

1. The population has a birth rate and death rate proportional to the population. Call the constants of proportionality β and α respectively. We assume $\beta - \alpha > 0$.
2. In addition, there is a death rate from the disease proportional to the contagious populations. Call this constant α_d.
3. Offspring of ill mothers are healthy.
4. The rate of infections is equal to bSC for some constant b.

1. Argue that changes in population over the course of the disease can be modelled by

$$S' = (\beta - \alpha)S + \beta C - bSC$$
$$C' = bSC - (\alpha_d + \alpha)C$$

subject to the initial populations S_0 and C_0. This is the system of equations on p. 245 of the text.

2. Now let P be the total population and show

$$P' = (\beta - \alpha)P - \alpha_d C$$
$$C' = b(P - C)C - (\alpha_d + \alpha)C$$

subject to the initial populations P_0 and C_0. This is often easier to work with.

3. Scale the last system by setting

$$P = \frac{\alpha_d}{b}x \quad C = \frac{\alpha_d}{b}y \quad t = \frac{1}{\alpha_d}s.$$

Your new equations should read

$$\frac{dx}{ds} = cx - y$$
$$\frac{dy}{ds} = y(x - y - r)$$

for $c = (\beta - \alpha)/\alpha_d$ and $r = (\alpha_d + \alpha)/\alpha_d$. Discuss how lowering the birth rate or raising the death rate affects the constants c and r. We will also always assume $x \geq y$. Does this make sense? Find the interesting equilibrium of this equation.

4. (Maple) We will explore what happens for various values of $c > 0$. Take $r = 2$ and $c = 1.5$. Take a number of number of initial conditions with $y = 2$ and x between 2 and 15. Graph x versus y and both x and y versus s. Interpret what you see. Where is the equilibrium in the x versus y picture?

5. (Maple) Now change c to $c = 0.25$ are repeat. Now take initial conditions of $x = 7$ and various values of y between 1 and 7.

6. Use a Jacobian argument to show that the equilibrium is stable if $c < 1$ and unstable if $c > 1$. Compare this to your answers to 4 and 5. What happens if $c = 1$?

7. (Maple) Use the data on the middle of p. 246 (above figure 8.7.1) to find values for c and r. Again graph x versus y and both x and y versus s for various initial conditions. You should see the population "bursts". Is the equilibrium stable?

8. (Partly Maple) Based on the data of problems 4 and 5 argue that proportional harvesting (called "culling" on pp. 246–247) would be a good way to control the population. If the harvesting rate κ show that this changes c and r to

$$c = \frac{\beta - \alpha - \kappa}{\alpha_d} \quad \text{and} \quad r = \frac{\alpha_d + \alpha + \kappa}{\alpha_d}$$

respectively. Now c can be negative. Take $c = -1$, $r = 2$ and various initial conditions with $x = 7$ to explore what happens.

9. A more humane way to limit the spread of the disease is through vaccination. This adds a third group to the population, the set Z of vaccinated members, so that $S+C+Z=P$. Assume that vaccination is proportional to the susceptible population with some proportionality constant γ. Show that the new differential equations become

$$\begin{aligned} P' &= (\beta - \alpha)P - \alpha_d C \\ C' &= b(P - C - Z)C - (\alpha_d + \alpha)C \\ Z' &= \gamma(P - C - Z) - \alpha Z. \end{aligned}$$

10. (Maple) Take the possum population of p. 246 and graph P and C versus time before and after vaccination ($\gamma = 0.33$) and before and after cullings ($\kappa = 0.33$). Take the initial conditions of Figure 8.7.5. Compare your answers.

11. Suppose your goal is limit the number of ill member of the population as quickly as possible. Is it better to vaccinate, cull, or do nothing?

As always, end your discussion with a summary of what you have discovered.