MATH 360-1, Fall 2008 **MENU Applied Analysis**

Test 2

1. Consider the second order ODE with $b > 0$ but very near zero:

$$y'' + 2by' + 4y = 4\cos(2t).$$

a.) Find the general solution. The solution to the associated homogeneous equation is

$$y_h = c_1 e^{-bt}\cos(at) + c_1 e^{-bt}\sin(at)$$

with $a = \sqrt{4 - b^2}$.

b.) What happens to your solution as $b \to 0$? Why would you expect this?

2. Consider the system of equations

$$x' = -y$$
$$y' = -4x$$

where differentiation is with respect to time.

a.) Find the general solution of this system.

b.) Draw the phase plane in all four quadrants. Show where $x' = 0$, where $y' = 0$ and any solutions that travel in straight lines. Sketch in a few solutions to show long term behavior.

3. The following system of differential equations governs two competing species.

$$x' = x(2 - x - y)$$
$$y' = y(4 - y - 3x)$$

a.) Draw and analyze the first-quadrant phase plane, showing where $x' = 0$, where $y' = 0$ and the behavior of the direction fields in various parts of the first quadrant. You should find and clearly label four different such areas.

b.) Find the equilibrium solution with $x \neq 0 \neq y$. Decide if it is stable, unstable, or a saddle point.

4. A hardened spring exerts a force of $k_1 x + k_2 x^3$ where x is the extension of the spring and k_1 and k_2 are positive constants.

a) Assume the spring is suspended from the ceiling and has mass m attached. Write an equation for the forces at equilibrium.

b) Write a differential equation describing the vertical displacement of the mass when the system is in motion.

c) Show that the equilibrium solution is a center.

Answers

1a.) The particular solution is $\frac{1}{b}\cos(2t)$.
b.) The particular solution achieves arbitrarily large values. If $b = 0$, $\cos(2t)$ is a solution of the associated homogeneous.

2. a.) $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t}$.
b.) Except when $c_2 = 0$, all solutions go asymptotically to the line $y = -2x$.

3b.) The equilibrium $x = 1 = y$ is a saddle point.

4. a) $mg = k_1 L + k_2 L^3$ where L is the extension of spring when at rest.
b.) $my'' + (k_1 + 3L^2 k_2)y + 3Lk_2^2 y^2 + k_2 y^3 = 0$; the equilibrium is now at $y = y' = 0$.
c) Rewrite the differential equation as

$$\begin{aligned} u' &= v \\ v' &= -\frac{(k_1 + 3L^2 k_2)}{m} u - \frac{3Lk_2^2}{m} u^2 - \frac{k_2}{m} u^3 \end{aligned}$$

with $u = y$ and $v = y'$. The Jacobian of this system at $u = 0$ and $v = 0$ is

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{(k_1 + 3L^2 k_2)}{m} & 0 \end{pmatrix}$$

which has purely imaginary roots.