A groupoid is a category $\mathcal{G}$ so that (1) the collection of objects is a set and (2) every morphism $f : x \to y$ in $\mathcal{G}$ has a two-sided inverse. For example, if $X$ is a topological space, then $\Pi_X$ is the groupoid with objects the points in $X$ and the morphisms the collection $\Pi(x, y)$ of paths modulo endpoint preserving homotopy. This is the fundamental groupoid. Let’s write

$$\text{Aut}_\mathcal{G}(x) = \mathcal{G}(x, x) = \text{Hom}_\mathcal{G}(x, x)$$

for the set of self maps if $x \in \mathcal{G}$. This is a group or self-isomorphisms, or automorphisms.

1. a) Show there is a category $\text{Grpd}$ with objects all groupoids and morphisms the functors between groupoids.
   b) Show that the assignment $X \mapsto \Pi_X$ is a functor from spaces to groupoids.

If $F, G : \mathcal{C} \to \mathcal{D}$ are two functors, a natural transformation from $\eta : F \to G$ assigns to each $X \in \mathcal{C}$ a morphism $\eta_X : FX \to GX$ in $\mathcal{D}$ so that for every morphism $f : X \to Y$ in $\mathcal{C}$ there is a commutative diagram

$$
\begin{array}{ccc}
FX & \xrightarrow{\eta_X} & GX \\
F(f) \downarrow & & \downarrow G(f) \\
FY & \xrightarrow{\eta_Y} & GY.
\end{array}
$$

Two groupoids $\mathcal{G}$ and $\mathcal{H}$ are equivalent if there are functors

$$
\mathcal{G} \xrightarrow{F} \mathcal{H} \xleftarrow{G}
$$

and natural transformations $1_{\mathcal{G}} \to GF$ and $1_{\mathcal{H}} \to FG$.

2.a) Show that if $F : \mathcal{G} \to \mathcal{H}$ is part of an equivalence of groupoids, then $F$ induces an isomorphism of groups

$$F_* : \text{Aut}_\mathcal{G}(x, x) \to \text{Aut}_\mathcal{H}(Fx).$$
b) Suppose we have homotopic maps $f, g : X \to Y$. Show that a choice of homotopy defines a natural transformation $\Pi(f) \to \Pi(g)$.

c) Suppose spaces $X$ and $Y$ are homotopy equivalent (with no assumption on base points), then $\Pi_X$ and $\Pi_Y$ are equivalent.