MATH 460, Winter 2008

Algebraic Topology

Final Exam Review

1. Find a triangulation for $S^1 \vee S^2$ and give explicit cycle representatives for generators of $H_i(S^1 \vee S^2)$, $i = 0, 1, 2$.

2. Let A_i, $i \geq 1$, be a sequence of finitely generated abelian groups. Find a path-connected space X so that $H_i X \cong A_i$ for $i \geq 1$.

3. Give an example of two spaces with the same homology which are not homotopy equivalent.

4. Suppose $U_1, \ldots, U_n$ is finite open cover of a space X so that all intersections

$$U_{i_1} \cap \cdots \cap U_{i_s}$$

are contractible for $1 \leq s < n$. Find a relation between $H_* X$ and

$$H_*(U_1 \cap \cdots \cap U_n).$$

If it makes you more comfortable, you can assume this last intersection is non-empty, but it's not necessary to do so.

5. Let X be the subspace of $\mathbb{R}^3$ obtained by taking the union of two spheres of radius 2, one centered at $(3/2, 0, 0)$ and the other at $(-3/2, 0, 0)$. Find a CW structure for X, write down the CW chain complex, and compute its homology.

6. Let $f : \mathbb{R}\mathrm{P}^2 \to S^2$ be the map which collapses the subspace $\mathbb{R}\mathrm{P}^1 \subseteq \mathbb{R}\mathrm{P}^2$. Calculate this map in homology with $\mathbb{Z}$-coefficients and with $\mathbb{Z}/2$-coefficients. Use your result to show that the splitting in the universal coefficient sequence cannot be made natural.

7.i) Let $\mathbb{Z}/p^\infty \subseteq \mathbb{Q}/\mathbb{Z}$ be the p-torsion for some prime p. Suppose we have a space X so that

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}/p^\infty, & i = 1; \\ 0, & \text{else.} \end{cases}$$

Calculate $H_*(X, \mathbb{Z}/p)$. You may want use that there is a short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^\infty \xrightarrow{\times p} \mathbb{Z}/p^\infty \to 0.$$

ii.) Write $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z})$ as a functor of the abelian group A.

8. If M_1 and M_2 are two closed (compact and without boundary) n-manifolds, then their connected sum $M_1 \sharp M_2$ is the space obtained as follows: from each M_i remove a small open n-ball, then glue the resulting spaces together along the resulting boundary $n-1$-spheres. (You might get different results depending on how you identify the spheres, but that won't matter for this problem.) Show that If M_1 and M_2 are two manifolds, then there is an equality of Euler characteristics

$$\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - 2.$$

What can you say about 3-manifolds? Or n-manifolds? You can expand this problem: let K be a simplicial complex which is the union of two subcomplexes A and B. Write $\chi(K)$ as formula of $\chi(A)$ and $\chi(B)$ and whatever data you need.

9. Let $C_\bullet$ be a chain complex over the rationals. Prove

$$\sum_i (-1)^i \dim C_i = \sum_i (-1)^i \dim H_i C_\bullet.$$

10. Prove or disprove the following statements. If you disprove it, then fix it if possible.

i.) Let $f : X \to X$ be a self-map of a contractible space. Then f has a fixed point.

ii.) If $\mathbb{R}\mathrm{P}^n$ is a topological group, then n is even.

iii.) If $f : X \to X$ has a fixed point, then the Lefschetz number of f is non-zero.

11. Verify that the function

$$f(z_1, \ldots, z_{2k}) = (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3, \ldots, \bar{z}_{2k}, -\bar{z}_{2k-1})$$

from $\mathbb{C}^{2k}$ to itself defines a function $g : \mathbb{C}\mathrm{P}^{2k-1} \to \mathbb{C}\mathrm{P}^{2k-1}$ to itself with no fixed points. Reconcile this with Lefschetz fixed point theorem.

12. Let $z \in H_1(S^1)$ be a generator and let (X, y) be a path-connected and based space. Define a function $g : \pi_1(X, y) \to H_1(X)$ by sending a loop ϕ to $\phi_* z$. Let $\pi_1^{ab}(X)$ be the abelianization of $\pi_1(X, y)$. Prove

i.) The group $\pi_1^{ab}(X)$ does not depend on y.

ii.) The g defines a homomorphism $h : \pi_1^{ab}(X) \to H_1(X)$. (This homomorphism is in fact an isomorphism.)

iii.) Let X be a CW complex and suppose $H_1(X)$ has all p-torsion for some prime $p \neq 2$. Show every map $X \to \mathbb{R}\mathrm{P}^\infty$ is contractible.

13. Let $f : A \to X$ and $g : A \to Y$ be two maps. Define $C(f, g)$ to be the quotient space of the disjoint union $X \sqcup A \times [0, 1] \sqcup Y$ with $f(a) \sim (a, 0)$ and $(a, 1) \sim g(a)$. Describe the homology of $C(f, g)$ by a long exact sequence. Then calculate this homology when $f, g : S^n \to S^n$ are maps of degree d_1 and d_2 respectively.