MATH 460, Winter 2008

Algebraic Topology

Problem Set 1

1. Find some triangulation K of the two-sphere S^2 and calculate $H_2(K)$. "Find a triangulation" means find a simplicial complex K whose underlying topological space is S^2.

2. A map of simplicial complexes $f : K \to L$ is a function which sends vertices to vertices and is linear on each simplex. If K and L are ordered simplicial complexes, then f may not preserve the ordering, however f still induces a map on chain complexes as follows.

Let $[x_1, \ldots, x_n]$ be a simplex of K with $x_1 < \cdots < x_n$. If $f(x_i) \neq f(x_j)$ for $i \neq j$, then rewriting $f(x_1), \ldots, f(x_n)$ in order as

$$f(x_{i_1}) < \ldots < f(x_{i_n})$$

defines a permutation τ of $\{1, \ldots, n\}$. Define

$$f_\sharp : C_n(K) \to C_n(L)$$

by

$$f_\sharp([x_1, \ldots, x_n]) = \text{sign}(\tau)[f(x_{i_1}), \ldots, f(x_{i_n}].$$

If the vertices of a simplex get collapsed, the $f_\sharp$ sends that simplex to 0. You could try to actually prove this is a chain map, but it's somewhat finicky and we'll have better ways to do this later; thus, in the spirit of getting on with computations, let's assume this fact.

a.) Let $a : S^n \to S^n$ be given by $a(x) = -x$. Find a triangulation K of S^1 which respects a and compute the induced map on $H_1(K)$.

b.) Do the same as in part b for S^2, but now for H_2. The answer will be different.

3. Let $f, g : C \to D$ be two morphisms of chain complexes. Then a *chain homotopy* T from f to g is a set of homomorphisms

$$T : C_n \to D_{n+1}$$

so that

$$\partial T + T\partial = f - g.$$

If $n = 0$ this simply says $\partial T = f - g$. What this has to do with real homotopies we'll see later.

a.) Prove that if there is chain homotopy from f to g, then $f_* = g_* : H_*C \to H_*D$.

b.) We say f and g are chain homotopy equivalent if we can find a chain homotopy from f to g. Show that this is, indeed, an equivalence relation on the set of chain maps from $C \to D$.

c.) Define $[C, D]$ to be the chain homotopy equivalence class of chain maps from $C \to D$. (Warning: this definition is usually used only if C_n is a free abelian group for all n.) Show that composition descends to a composition pairing

$$\begin{aligned} [D, E] \times [C, D] &\longrightarrow [C, E] \\ ([g], [f]) &\mapsto [g \circ f] \end{aligned}$$