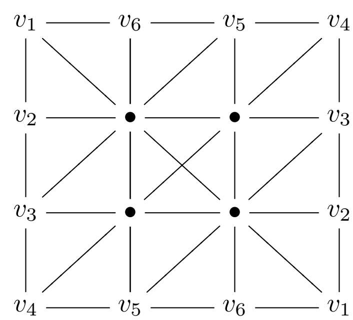
MATH 460, Winter 2008

Algebraic Topology

Problem Set 2

1. Let $\mathbb{RP}^2$ denote quotient space $S^2/\sim$ where $\sim$ is the equivalence relation with $x \sim x$ or $x \sim -x$. A triangulation of $\mathbb{RP}^2$ is given below. (The center crossing 1-simplices meet in a vertex, which I didn't want to draw.) Compute its homology.



2.i) Let $\mathbb{Z}$ denote the chain complex which has a copy of the integers in degree 0 and the zero group in every other degree. If X is a space and $C_\bullet(X)$ its singular complex, define

$$\epsilon : C_0(X) \to \mathbb{Z}$$

to be the unique homomorphism which is takes a singular zero simplex to 1. Show that this gives a morphism of chain complexes.

ii.) Let $\tilde{C}_\bullet(X)$ be the kernel of ϵ and the reduced homology $\tilde{H}_*(X)$ the homology of this chain complex. Show $H_n(X) = \tilde{H}_n(X)$ for $n > 0$ and show that a choice of basepoint $x_0 \in X$ defines an isomorphism

$$H_0(X) \cong \tilde{H}_0 X \oplus \mathbb{Z}.$$

iii.) Show there is a Mayer-Vietoris sequence in reduced homology.

3. Let X be a space and SX its suspension; thus SX is $X \times [-1, 1]$ with $X \times \{1\}$ collapsed to one point and $X \times \{-1\}$ to another. Show that for all n, $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$. Use this to give another calculation of the homology of a sphere.

4.i) Let $f : X \to Y$ be a continuous map. Let CX be the quotient space of $X \times [0, 1]$ with $X \times \{1\}$ collapsed to a point – this is the cone on X. The mapping cone of f is the quotient space $Y \cup_f CX$ with $f(x) \sim x \times \{0\}$. Let $i : Y \to C(f)$ be the inclusion. Show that there is a long exact sequence

$$\cdots \to H_n X \xrightarrow{f_*} H_n Y \xrightarrow{i_*} \tilde{H}_n C(f) \to H_{n-1} X \to \cdots$$

ii) Show $H_n(C(f), Y) \cong \tilde{H}_{n-1}(X)$.

5. Use problem 4 to give another calculation of $H_*\mathbb{R}\mathrm{P}^2$.

6. Prove the Yoneda Lemma. Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$. Then X defines a functor $F_X : \mathcal{C} \to \mathbf{Sets}$ by

$$F_X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

Now suppose $T : \mathcal{C} \to \mathbf{Sets}$ is any functor. Then if $f : F_X \to T$ is a natural transformation the image of the identity in $\mathrm{Hom}_{\mathcal{C}}(X, X)$ defines an element

$$f(\mathrm{id}_X) \in T(X).$$

Show that this assignment gives a one-to-one correspondence between the natural transformations $F_X \to T$ and the elements of $T(X)$.

This can be adapted to abelian groups. Let $\mathbf{Ab}$ be the category of abelian groups. Then we get a functor $G_X : \mathcal{C} \to \mathbf{Ab}$ be setting

$$G_X(Y) = \mathbb{Z}\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

where $\mathbb{Z}[-]$ is the free abelian group functor. If $A : \mathcal{C} \to \mathbf{Ab}$ is any functor to abelian groups, then the natural homomorphisms $G_X \to A$ are in one-to-correspondence with $A(X)$.