MATH 460, Winter 2008

Algebraic Topology

Problem Set 3

1. Let X be a topological space and $C_\bullet X$ the singular chains on X. Suppose

$$f, g : C_\bullet X \rightrightarrows C_\bullet X$$

are two natural chain maps which agree in degree zero. Show that they must be chain homotopic. Then show that, up to chain homotopy, that the only natural chain maps are f_n, $n \in \mathbb{Z}$ where $f_n(y) = ny$.

Here f *natural* means that there is an $f = f_X$ for each topological space X and that for any continuous map $\phi : X \to Y$ there is a commutative diagram

$$\begin{array}{ccc} C_\bullet X & \xrightarrow{f_X} & C_\bullet X \\ {\scriptstyle \phi_\sharp}\downarrow & & \downarrow{\scriptstyle \phi_\sharp} \\ C_\bullet Y & \xrightarrow[f_Y]{} & C_\bullet Y. \end{array}$$

2. The complex projective space $\mathbb{C}\mathrm{P}^n$ is the set of all subspaces of complex dimension 1 (lines) in $\mathbb{C}^{n+1}$. If we let $S^{2n+1} \subseteq \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ be the unit sphere, then the circle $S^1 \subseteq \mathbb{C}$ acts on S^{2n+1}. Show that there is an isomorphism

$$S^{2n+1}/S^1 \cong \mathbb{C}\mathrm{P}^n.$$

This gives a topology to $\mathbb{C}\mathrm{P}^n$. If $q : S^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ is the quotient map, show that

$$\mathbb{C}\mathrm{P}^n \cong \mathbb{C}\mathrm{P}^{n-1} \cup_q D^{2n}.$$

Define $\mathbb{C}\mathrm{P}^\infty = \cup\, \mathbb{C}\mathrm{P}^n$. Give $\mathbb{C}\mathrm{P}^n$, $1 \leq n \leq \infty$ a CW structure and compute the associated CW chain complex and its homology.

3. Give CW decompositions for the following subspaces of $\mathbb{R}^3$; write down the associated chain complex as well.

1. The space X obtained as the union of the two spheres of radius 1, the first about $(1, 0, 0)$, the second about $(-1, 0, 0)$.

2. The space Y obtained as the union of standard unit sphere and a disk of radius 1 in the xy-plane.

4. A Δ-complex is a collection of sets X_n, $n \geq 0$, equipped with maps $d_i : X_n \to X_{n-1}$ so that $d_i d_j = d_{j-1} d_i$ for $i < j$. The geometric realization quotient space

$$|X| = \coprod_n X_n \times \Delta^n / \simeq$$

where $\simeq$ is the smallest equivalence relation with

$$(d_i(x), t) \simeq (x, d^i(t)).$$

Show that $|X|$ is a CW complex with associated chain complex given by C_n the free abelian group on X_n and

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : C_n|X| \longrightarrow C_{n-1}|X|.$$

5. Show how an ordered simplicial complex gives a Δ-complex and the chain complex so obtained is the chain complex of an ordered simplicial complex.

6. Explain how the following is a picture of a Δ-complex. What is the realization? (The triangles are filled.) Compute the homology using the resulting chain complex.