MATH 470 Winter 2007

Graduate Algebra

Final Project

The main point of this collection of problems is to explore Artin's Theorem below, which gives some information on which representations can be obtained by inducing up from subgroups.

1. Calculate the representation rings $R(A_4)$ and $R(S_4)$.

2. Let D_n be the dihedral of rigid motions of a regular n-gon; it is of order $2n$. Let $C_n \subseteq D_n$ be the cyclic subgroup be the cyclic subgroup generated by the rotation by $2\pi/n$. If γ is a primitive nth root unity, let $V_k = V_{\gamma^k}$, $1 \leq k < n$ be the irreducible representations of C_n. Calculate the characters of the induced representations

$$W_k = \mathrm{Ind}_{C_n}^{D_n}(V_k).$$

Decide when $W_k = W_m$. Calculate the character table of D_n.

In what follows we let $\mathcal{A}$ denote a set of subgroups of a finite group G. Examples of $\mathcal{A}$ could be the set of cyclic subgroups or the set of abelian subgroups, both of which have the additional properties that $\mathcal{A}$ is closed under conjugation and passage to subgroups.

Theorem 1 (Artin's Theorem). *Let $\mathcal{A}$ be a set of subgroups of a finite group G. Let*

$$\mathrm{Ind} : \oplus_{H\in\mathcal{A}} R(H) \longrightarrow R(G)$$

be the homomorphism defined by induction up from the subgroups. Then the following are equivalent:

1. *$G = \cup\, gHg^{-1}$ where $g \in G$ and $H \in \mathcal{A}$;*
2. *the cokernel if* Ind *is finite.*

You will be asked to prove this below, but first here are some examples.

3. Rephrase the second condition of the theorem as follows: for each character χ of G there are virtual characters $\chi_H \in R(H)$, $H \in \mathcal{A}$ and an integer N so that

$$N\chi = \sum_{H\in\mathcal{A}} \mathrm{Ind}_H^G(\chi_H).$$

4. Let $\mathcal{A}$ be the set of cyclic subgroups. Show that if n is odd, then the map Ind is onto for $G = D_n$.

5. Again let $\mathcal{A}$ be the set of cyclic subgroups and let $G = A_4$. Show that $\chi \in R(A_4)$ is in the image of Ind if and only if $\chi(1)$ is even. Show that no irreducible character is in the image. What happens if you replace $\mathcal{A}$ be the set of abelian subgroups?

6. Prove (2) $\implies$ (1) in Artin's Theorem. To do this make, first make the following observation: let X be the union of conjugates of the subgroups in $\mathcal{A}$. Then each class function of the form

$$\sum_{H\in\mathcal{A}} \mathrm{Ind}_H^G(\chi_H)$$

vanishes for $y \in G - X$.

7. Note that $\mathbb{C} \otimes R(G) = \mathrm{map}(G/\mathrm{conj}, \mathbb{C})$. Use Frobenius reciprocity to show that the $\mathbb{C}$-linear dual of

$$\mathrm{Ind}_H^G : \mathbb{C} \otimes R(H) \longrightarrow \mathbb{C} \otimes R(G)$$

is the restriction

$$\mathrm{Res} : \mathbb{C} \otimes R(G) \longrightarrow \mathbb{C} \otimes R(H).$$

8. Prove (1) $\implies$ (2) in Artin's Theorem. To do this make the observation that it is sufficient to show

$$\mathbb{C} \otimes \mathrm{Ind} : \oplus_{H\in\mathcal{A}} \mathbb{C} \otimes R(H) \longrightarrow \mathbb{C} \otimes R(G)$$

is onto, for then $\mathbb{Q} \otimes \mathrm{Ind}$ will also be onto. Using the previous result, show that $\mathbb{C} \otimes \mathrm{Ind}$ is onto if and only if the sum of restriction maps

$$\mathrm{Res} : \mathbb{C} \otimes R(G) \longrightarrow \oplus_{H\in\mathcal{A}} \mathbb{C} \otimes R(H)$$

is one-to-one.

9. (Extra credit) Let $\mathcal{A}$ be a set of subgroups closed under . Artin's theorem implies that

$$\mathbb{Q} \otimes \mathrm{Ind} : \oplus_{H\in\mathcal{A}} \mathbb{Q} \otimes R(H) \longrightarrow \mathbb{Q} \otimes R(G)$$

is onto. What is the kernel? Here's two observations.

1. If $H_0 \subseteq H_1$ and $V \in R(H_0)$, then

$$\mathrm{Ind}_{H_0}^G(V) = \mathrm{Ind}_{H_1}^G \mathrm{Ind}_{H_0}^{H_1}(V);$$

2. If V is an H-representation and $g \in G$, define a representation V_g over gHg^{-1} by setting V_g to be the same vector space as V but with the action of $h \in gHg^{-1}$ given by

$$h \cdot x = (g^{-1}hg)x.$$

Then $\mathrm{Ind}_{gHg^{-1}}^G(V_g) = \mathrm{Ind}_H^G(V)$.