1. Let $R$ be a principal ideal domain and $a, b \in R$. Set $c$ be the greatest common divisor of $a$ and $b$. Prove, with all details, that

$$R/(a) \otimes_R R/(b) \cong R/(c).$$

2. Let $D_4$ be the dihedral group with 8 elements with generators $\tau$ of order 4 and $\sigma$ of order 2. Then $D_4$ has 4 distinct isomorphism classes of 1-dimensional complex representations $V(1,1), V(-1,1), V(1,-1)$ and $V(-1,-1)$ where, for example, if $x \in V(1,-1)$ then $\tau(x) = x$ and $\sigma(x) = -1$. There is also an irreducible 2-dimensional complex representation $W \cong \mathbb{C}^2$ with

$$\tau(x, y) = (-y, x) \quad \text{and} \quad \sigma(x, y) = (x, -y).$$

Show that there is an isomorphism of representations

$$W \otimes_{\mathbb{C}} W \cong V(1,1) \oplus V(1,-1) \oplus V(-1,1) \oplus V(-1,-1).$$

3. Let $N$ be a fixed $R$-module. Then we have two functors from $R$-modules to $R$-modules given by

$$M \mapsto M \otimes_R N$$
$$M \mapsto \text{Hom}_R(N, M).$$

Given a homomorphism $f : M_1 \otimes_R N \to M_2$, define a homomorphism $g : M_1 \to \text{Hom}_R(N, M_2)$ as follows: if $x \in M_1$, set $g(x) : N \to M_2$ to be $g(x)(y) = f(x \otimes y)$. Show this defines an isomorphism

$$\text{Hom}_R(M_1 \otimes_R N, M_2) \cong \text{Hom}_R(M_1, \text{Hom}_R(N, M_2)).$$

4. Answer only: find the splitting fields of the following polynomials.

i.) $x^4 - 4 \in \mathbb{Q}[x]$.

ii.) $x^5 - 1 \in \mathbb{Q}[x]$.

iii.) $x^3 - 10 \in \mathbb{Q}(\sqrt{2})[x]$.

iv.) $x^4 - 4x^2 - 1 \in \mathbb{Q}[x]$.

5. Let $F \subseteq E$ be a field extension with $E$ algebraic over $F$. Let $a_1, \ldots, a_n \in E$. Prove that if $\sigma$ is an element of the Galois group of $F(a_1, \ldots, a_n)$ so that $\sigma(a_i) = a_i$ for all $i$, then $\sigma$ is the identity.
6. Let $F \subseteq \mathbb{E}$ be the splitting field of some polynomial $f(x) \in \mathbb{F}[x]$. Let $G$ be the Galois group. Prove that there is an injection

$$\Phi : G \longrightarrow S_n$$

where $n$ is the degree of $f(x)$.

Hint: $G$ acts on the roots of $f(x)$. This was Galois’s point of view.

7. Let $\mathbb{F}$ be a field of characteristic $p$, with $p > 0$.

i.) Define the Frobenius $\phi : \mathbb{F} \rightarrow \mathbb{F}$ to be the $p^{\text{th}}$ power map: $\phi(x) = x^p$. Prove $\phi$ is field homomorphism.

ii.) An algebraic extension $\mathbb{F} \subseteq \mathbb{E}$ of fields of characteristic $p > 0$ is called separable if whenever $a \in \mathbb{E}$ and $a^p \in \mathbb{F}$, then $a \in \mathbb{F}$. Suppose the Frobenius $\phi : \mathbb{F} \rightarrow \mathbb{F}$ is an isomorphism. Prove any algebraic extension of $\mathbb{F}$ is separable.

iii.) A field for which the Frobenius is an isomorphism is called perfect. Show any finite field is perfect.

iv.) Give an example of a non-separable extension.

8. This introduces another algebraic construction defined by a universal property. An example will be the Galois group of a not-necessarily finite algebraic extension of fields.

Suppose we have a set of groups $G_i$, $i \in I$, and for each pair $(i, j) \in I \times I$ a (possibly empty) set of homomorphisms

$$f_{i,j}^\alpha : G_i \longrightarrow G_j.$$ 

The inverse limit of this “diagram” of groups is a group $H$ so that

a.) $H$ is equipped with homomorphisms $\epsilon_i : H \rightarrow G_i$ so that $f_{i,j}^\alpha \circ \epsilon_i = \epsilon_j$ for all $i$, $j$, and $\alpha$;

b.) if $H'$ is any group equipped with homomorphisms $\epsilon_i'$ as in a.), there is a unique homomorphism $g : H' \rightarrow H$ so that $\epsilon_i \circ g = \epsilon_i'$.

Prove that the inverse limit exists and is unique up to isomorphism. We write $\lim G_i$ for this group.

Hint: For existence, consider the subgroup of $\prod G_i$ of elements $(g_i)$ so that $f_{i,j}^\alpha(g_i) = g_j$ for all $i$, $j$, and $\alpha$. 