

MATH 470 Winter 2007

Graduate Algebra

Problem Set 1

1. Let R be a principal ideal domain and $a, b \in R$. Set c be the greatest common divisor of a and b . Prove, with all details, that

$$R/(a) \otimes_R R/(b) \cong R/(c).$$

2. Let D_4 be the dihedral group with 8 elements with generators τ of order 4 and σ of order 2. Then D_4 has 4 distinct isomorphism classes of 1-dimensional complex representations $V(1, 1)$, $V(-1, 1)$, $V(1, -1)$ and $V(-1, -1)$ where, for example, if $x \in V(1, -1)$ then $\tau(x) = x$ and $\sigma(x) = -x$. There is also an irreducible 2-dimensional complex representation $W \cong \mathbb{C}^2$ with

$$\tau(x, y) = (-y, x) \quad \text{and} \quad \sigma(x, y) = (x, -y).$$

Show that there is an isomorphism of representations

$$W \otimes_{\mathbb{C}} W \cong V(1, 1) \oplus V(1, -1) \oplus V(-1, 1) \oplus V(-1, -1).$$

3. Let N be a fixed R -module. Then we have two functors from R -modules to R -modules given by

$$\begin{aligned} M &\mapsto M \otimes_R N \\ M &\mapsto \text{Hom}_R(N, M). \end{aligned}$$

Given a homomorphism $f : M_1 \otimes_R N \rightarrow M_2$, define a homomorphism $g : M_1 \rightarrow \text{Hom}_R(N, M_2)$ as follows: if $x \in M_1$, set $g(x) : N \rightarrow M_2$ to be $g(x)(y) = f(x \otimes y)$. Show this defines an isomorphism

$$\text{Hom}_R(M_1 \otimes_R N, M_2) \cong \text{Hom}_R(M_1, \text{Hom}_R(N, M_2)).$$

4. Answer only: find the splitting fields of the following polynomials.

i.) $x^4 - 4 \in \mathbb{Q}[x]$.

ii.) $x^5 - 1 \in \mathbb{Q}[x]$.

iii.) $x^3 - 10 \in \mathbb{Q}(\sqrt{2})[x]$.

iv.) $x^4 - 4x^2 - 1 \in \mathbb{Q}[x]$.

5. Let $\mathbb{F} \subseteq \mathbb{E}$ be a field extension with \mathbb{E} algebraic over \mathbb{F} . Let $a_1, \dots, a_n \in \mathbb{E}$. Prove that if σ is an element of the Galois group of $\mathbb{F}(a_1, \dots, a_n)$ so that $\sigma(a_i) = a_i$ for all i , then σ is the identity.

6. Let $\mathbb{F} \subseteq \mathbb{E}$ be the splitting field of some polynomial $f(x) \in \mathbb{F}[x]$. Let G be the Galois group. Prove that there is an injection

$$\Phi : G \longrightarrow S_n$$

where n is the degree of $f(x)$.

Hint: G acts on the roots of $f(x)$. This was Galois's point of view.

7. Let \mathbb{F} be a field a characteristic p , with $p > 0$.

i.) Define the Frobenius $\phi : \mathbb{F} \rightarrow \mathbb{F}$ to be the p^{th} power map: $\phi(x) = x^p$. Prove ϕ is field homomorphism.

ii.) An algebraic extension $\mathbb{F} \subseteq \mathbb{E}$ of fields of characteristic $p > 0$ is called *separable* if whenever $a \in \mathbb{E}$ and $a^p \in \mathbb{F}$, then $a \in \mathbb{F}$. Suppose the Frobenius $\phi : \mathbb{F} \rightarrow \mathbb{F}$ is an isomorphism. Prove any algebraic extension of \mathbb{F} is separable.

iii.) A field for which the Frobenius is an isomorphism is called *perfect*. Show any finite field is perfect.

iv.) Give an example of a non-separable extension.

8. This introduces another algebraic construction defined by a universal property. An example will be the Galois group of a not-necessarily finite algebraic extension of fields.

Suppose we have a set of groups G_i , $i \in I$, and for each pair $(i, j) \in I \times I$ a (possibly empty) set of homomorphisms

$$f_{i,j}^\alpha : G_i \longrightarrow G_j.$$

The *inverse limit* of this "diagram" of groups is a group H so that

a.) H is equipped with homomorphisms $\epsilon_i : H \rightarrow G_i$ so that $f_{i,j}^\alpha \circ \epsilon_i = \epsilon_j$ for all i, j , and α ;

b.) if H' is any group equipped with homomorphisms ϵ'_i as in a.), there is a unique homomorphism $g : H' \rightarrow H$ so that $\epsilon_i \circ g = \epsilon'_i$.

Prove that the inverse limit exists and is unique up to isomorphism. We write $\lim G_i$ for this group.

Hint: For existence, consider the subgroup of $\prod_i G_i$ of elements (g_i) so that $f_{i,j}^\alpha(g_i) = g_j$ for all i, j , and α .