1. Let $\mathbb{F}$ be a field of characteristic $p$ and let $\mathbb{F} \subseteq \mathbb{E}$ be a normal extension. Define $\mathbb{F}_\infty \subseteq \mathbb{E}$ to be the subset

$$
\mathbb{F}_\infty = \{ a \in \mathbb{E} \mid a^{p^n} \in \mathbb{F} \text{ for some } n \}.
$$

Prove that $\mathbb{F}_\infty$ is a field, then $\mathbb{F} \subseteq \mathbb{F}_\infty$ is purely inseparable and the $\mathbb{F}_\infty \subseteq \mathbb{E}$ is separable.

Hint: recall that we have shown that if $a$ is algebraic over $\mathbb{F}$, then the minimal polynomial can be written (over the algebraic closure of $\mathbb{F}$) as $h(x)^{p^n}$ for some separable polynomial $h(x)$. Show that $h(x) \in \mathbb{F}_\infty[x]$.

2. Continuing problem one, show that $\text{Gal}(\mathbb{E}, \mathbb{F}) = \text{Gal}(\mathbb{E}, \mathbb{F}_\infty)$. Conclude that the main theorem of Galois theory yields, for finite normal extensions, an anti-equivalence of between the lattice of subgroups of $\text{Gal}(\mathbb{E}, \mathbb{F})$ and the subfields $\mathbb{F}_\infty \subseteq \mathbb{E}_0 \subseteq \mathbb{E}$.

3. Let $\mathbb{F}$ be a field of characteristic $p$ and let $\mathbb{F} \subseteq \mathbb{E}$ be an algebraic extension. Let $\mathbb{F}_s$ be the set of elements in $\mathbb{E}$ which are the roots of separable polynomial.

i.) Prove that $\mathbb{F}_s$ is a field. As a hint, suppose $a, b \in \mathbb{F}_s$. What is the separable degree of $\mathbb{F}(a, b)$ over $\mathbb{F}$? What can you conclude about the separable degree of $\mathbb{F}(a + b)$?

ii.) Show that $\mathbb{F}_s \subseteq \mathbb{E}$ is purely inseparable. Hint: recall that we have shown that if $a$ is algebraic over $\mathbb{F}$, then the minimal polynomial can be written (over the algebraic closure of $\mathbb{F}$) as $g(x^{p^n})$ for some separable polynomial. Show that $g(x)$ is a separable polynomial.

4. Display the Galois correspondence between subfields and subgroups for the splitting fields of the following polynomials over the rationals.

i.) $x^3 - 2$;

ii.) $(x^2 - 2)(x^2 - 3)$;

iii.) $x^6 - 1$;

iv.) $x^8 - 1$. 

5. Let \( \mathbb{F} \) be a field and \( G \) a finite group. If \( V \) is an \( \mathbb{F} \)-vector space, let \( \text{map}(G, V) \) be the set of set functions from \( G \) to \( V \). (This is the product \( V^G \).) The set \( \text{map}(G, V) \) is a vector space using addition and scalar multiplication in the target.

i.) If \( g \in G \) and \( \phi \in \text{map}(G, V) \), define a new function \( g\phi \in \text{map}(G, V) \) by \( g\phi(x) = \phi(xg) \). Prove that this is a left action and \( \text{map}(G, V) \) is a representation.

ii.) Define \( \epsilon : \text{map}(G, V) \to V \) by \( \epsilon(\phi) = \phi(e) \). Show this map has the following universal property. Suppose \( W \) is any representation and \( f : W \to V \) is \( \mathbb{F} \)-linear, then there is a morphism \( g : W \to \text{map}(G, V) \) of representations so that \( \epsilon g = f \).

iii.) Show that there is an isomorphism of \( \mathbb{F}[G] \to \text{map}(G, \mathbb{F}) \) but this map depends on choices.

6. (The Normal Basis Theorem. See Lang, p. 312.) Let \( \mathbb{F} \subseteq \mathbb{E} \) be a finite Galois extension with Galois group \( G \). Since \( G \) acts on \( \mathbb{E} \) through \( \mathbb{F} \)-linear transformations \( \mathbb{E} \) is representation of \( G \) over \( \mathbb{F} \). The normal basis theorem, as usually phrased as \( \mathbb{E} \cong \mathbb{F}[G] \), but there is no choice-free isomorphism. To get a canonical answer, it turns out to be easier to identity \( \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \).

i.) Regard \( \mathbb{E} \) as an \( \mathbb{F} \) vector space and consider the representation \( \text{map}(G, \mathbb{E}) \) defined in the previous problem. Prove this an \( \mathbb{E} \)-algebra using multiplication in the target.

ii.) Define a \( G \) action on \( \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \) by \( g(x \otimes y) = x \otimes gy \). Let \( g : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \to \text{map}(G, \mathbb{E}) \) be the morphism of representations (5ii) so that \( \epsilon g : \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \to \mathbb{E} \) is multiplication. Show that this is an \( \mathbb{E} \)-algebra homomorphism that commutes with the \( G \) actions.

iii.) Show that is isomorphism. Conclude that \( \mathbb{E} \otimes_{\mathbb{F}} \mathbb{E} \cong \mathbb{E}[G] \) as a representation.