MATH 470 Winter 2007

Graduate Algebra
Problem Set 3

1. Complete the last problem of the previous problem set (no. 6iii).

2. Fix a ring R and prove the following are equivalent:

a.) R is semi-simple;

b.) Every left R module is projective;

c.) Every left R module is injective.

Relevant definitions are in Lang, almost any other algebra book, or on the web for that matter.

3. Let C_p be the cyclic group with p elements, p a prime. he ring $\mathbb{F}_p[C_p]$ is not semi-simple or, put another way, not every $\mathbb{F}_p$ representation of C_p is a direct sum of irreducible representations. Classify, up to isomorphism, all two dimensional $\mathbb{F}_p$ representations W of C_p so that

i.) W has a subrepresentation W_0 isomorphic to the trivial representation; and,

ii.) W/W_0 is isomorphic to the trivial representation.

Which are not a direct sum of trivial modules?

4. Prove that a commutative ring is semi-simple if and only if it is a finite product of fields.

5. Find the character tables of the following groups. These tables can be found in books, so provide a narrative of how you got your answers.

a.) D_5 – the dihedral group of order 10.

b.) D_6

c.) A_4, the alternating group in S_4.

Over

6. Let $H \subseteq G$ be a subgroup and V a representation of H. Define the induced representation V by

$$V = \mathrm{map}_H(G, W)$$

be the set of functions from G to W commuting with action of H. This is G-representation with $(g\phi)(x) = \phi(xg)$. The map $W \to V$ sending w to ϕ_w with

$$\phi_w(x) = \begin{cases} xw, & x \in H; \\ \\ 0, & \text{else} \end{cases}$$

is an injection of H representations. We confuse W with its image.

a.) Verify the claims in the previous paragraph.

b.) Let $x \in G$ and let xW be the evident translate of W. Show that $xW = yW$ if and only of $xH = yH$. If H is normal show xW is a sub-H-representation.

c.) Show $V = \oplus_{G/H}\, xW$ as vector spaces.

d.) Compute the character of V in terms of the character of W.

e.) Write the two-dimensional irreducible representations of D_5 as induced representations.