Vladimir Voevodsky, in a recent talk on Univalent Foundations (UF), argued that pure mathematics needs good proof assistants:

A technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail. [...] Mathematical research currently relies on a complex system of mutual trust based on reputations.

We also need elegant mathematical proofs, and thus readable formal proofs. I work in HOL Light, where John Harrison has promoted readability. I owe also Freek Wiedijk (miz3), Marco Maggesi, Petros Papapanagiotou, Vincent Aravantinos (Q-module) and Mark Adams. I designed readable.ml to be essentially miz3.ml + tactics (REWRITE\_TAC etc.). Many other people helped me learn HOL Light, especially Rob Arthan, Michael Norrish, and Vladimir Voevodsky.
Readable formal proofs are often discussed with the words *declarative* (write out the whole argument) and *procedural* (tell to the computer to prove the theorem), but that’s the wrong dichotomy. Better terms are *readability* (understand the formal proof without executing it) and *power* (automate the proof). But readability and power often are compatible goals. Brevity, and therefore automating tedious details, can aid readability.

I think the terminology problem was caused by Mizar, which has the great advantage of being relatively easy to use, learn and read, but these are largely (important) interface issues. The non-interface feature of Mizar is that it was designed to be declarative, via an interesting mathematical formula which forces (what’s calculated to be) small inference leaps.

Readable.ml is designed to be a powerful version of Freek’s miz3.ml: combine the miz3 interface with the power of tactics.
I’ll give examples from Hilbert axiomatic geometry and point set topology, 3500 lines of readable.ml HOL Light code each. The Hilbert code formalizes the plane geometry part of a 26 page paper I wrote to teach Geometry to my son, who’s practically a coauthor (http://www.math.northwestern.edu/~richter/hilbert.pdf). Geometry is the one high school math class involving mathematical proofs, and the proofs, which go back to Euclid, are rarely rigorous. It was very helpful to write formal proofs (originally in miz3) to check that my proofs were correct.

John Harrison has 20,000 lines of HOL Light code for point set topology in Euclidean space, and he suggested I generalize his results to arbitrary topological spaces.

I’d like to add that John really got me started in HOL Light, writing the first 100 lines of my first miz3 code porting some Mizar code I’d written for Tarski axiomatic geometry.
let CONNECTED_DIFF_OPEN_FROM_CLOSED = prove
(‘!s t u:real^N->bool.
  s SUBSET t \ A t SUBSET u \ /
  open s \ A closed t \ A connected u \ A connected(t DIFF s)
  ==> connected(u DIFF s)’,
REPEAT STRIP_TAC THEN REWRITE_TAC[connected; NOT_EXISTS_THM] THEN
MAP_EVERY X_GEN_TAC ['v:real^N->bool'; 'w:real^N->bool'] THEN STRIP_TAC THEN
UNDISCH_TAC 'connected(t DIFF s:real^N->bool)' THEN SIMP_TAC[connected] THEN
MAP_EVERY EXISTS_TAC ['v:real^N->bool'; 'w:real^N->bool'] THEN
ASM_REWRITE_TAC[] THEN
REPLICATE_TAC 2 (CONJ_TAC THENL [ASM SET_TAC[]; ALL_TAC]) THEN
POP_ASSUM_LIST(MP_TAC o end_itlist CONJ) THEN
MAP_EVERY (fun t -> SPEC_TAC(t,t)) ['v:real^N->bool'; 'w:real^N->bool']
MATCH_MP_TAC(MESON[
  ‘(!v w. P v w ==> P w v) \ A (!w v. P v w \ A Q w ==> F)
  ==> !w v. P v w ==> ~(Q v) \ A ~ (Q w)’) THEN
CONJ_TAC THENL [SIMP_TAC[CONJ_ACI; INTER_ACI; UNION_ACI]; ALL_TAC] THEN
REPEAT STRIP_TAC THEN
FIRST_X_ASSUM(MP_TAC o GEN_REWRITE_RULE I [connected]) THEN SIMP_TAC[]
MAP_EVERY EXISTS_TAC ['v UNION s:real^N->bool'; 'w DIFF t:real^N->bool'
ASM_SIMP_TAC[OPEN_UNION; OPEN_DIFF] THEN ASM SET_TAC[]);;
ConnectedSubtopology

\[ \vdash \forall \alpha \ s. \ s \subseteq \text{topspace} \ \alpha \]

\[ \Rightarrow (\text{Connected (subtopology } \alpha \ s) \iff \\
\neg (\exists e_1 \ e_2. \ \text{open}_\alpha e_1 \wedge \text{open}_\alpha e_2 \wedge \\
s \subseteq e_1 \cup e_2 \wedge e_1 \cap e_2 \cap s = \emptyset \wedge \\
\neg (e_1 \cap s = \emptyset) \wedge \neg (e_2 \cap s = \emptyset))) \]

let ConnectedDiffOpenFromClosed = theorem ‘;

\[ \forall \alpha \ s \ t \ u. \ u \subseteq \text{topspace} \ \alpha \Rightarrow s \subseteq t \wedge t \subseteq u \wedge \\
\text{open}_\alpha s \wedge \text{closed}_\alpha t \wedge \text{Connected (subtopology } \alpha \ u) \wedge \\
\text{Connected (subtopology } \alpha \ (t - s)) \Rightarrow \text{Connected (subtopology } \alpha \ (u - s)) \]

proof

ONCE_REWRITE_TAC TAUT

[\forall a \ b \ c \ d \ e \ f \ g. \ (a \wedge b \wedge c \wedge d \wedge e \wedge f \Rightarrow g) \iff \\
(a \wedge b \wedge c \wedge d \Rightarrow \neg g \Rightarrow f \Rightarrow \neg e)];
intro_TAC \( \forall \alpha \, s \subseteq t \cup u \), uSubset, st tu sOpen tClosed;
\( t - s \subseteq \text{topspace} \alpha \) \( \land \) u - s \subseteq \text{topspace} \alpha \] by fol uSubset sOpen OPEN_IN_SUBSET tClosed closed_in SUBSET_DIFF
SUBSET_TRANS;
simplify uSubset - ConnectedSubtopology;
rewrite LEFT_IMP_EXISTS_THM;
intro_TAC \( \forall[v/e1] [w/e2]; \)
intro_TAC vOpen wOpen u.sDisconnected vwDisjoint
vNonempty wNonempty;
rewrite NOT_EXISTS_THM;
intro_TAC t.sConnected;
\( t - s \subseteq v \cup w \) \( \land \) v \cap w \cap (t - s) = \emptyset \] by set tu u.sDisconnected vwDisjoint;
v \cap (t - s) = \emptyset \lor w \cap (t - s) = \emptyset \] by fol t.sConnected vOpen wOpen -;

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case_split vEmpty | wEmpty by fol -;
 suppose v \cap (t - s) = \emptyset;
 exists_TAC w \cup s; exists_TAC v - t;
 simplify vOpen wOpen sOpen tClosed OPEN_IN_UNION OPEN_IN_DIFF;
 set st tu u sDisconnected vEmpty vwDisjoint wNonempty vNonempty;
 end;
 suppose w \cap (t - s) = \emptyset;
 exists_TAC v \cup s; exists_TAC w - t;
 simplify vOpen wOpen sOpen tClosed OPEN_IN_UNION OPEN_IN_DIFF;
 set st tu u sDisconnected wEmpty vwDisjoint wNonempty vNonempty;
 end;
 qed;
Isn’t that a nice proof? We’re supposed to prove that if $u$ connected and $t – s$ connected, then $u – s$ connected. So we assume that $u – s$ is disconnected and $t – s$ is connected, and we must prove that $u$ is disconnected. So let $v$ and $w$ be open sets of $u$ which form a disconnection of $u – s$. Since $t – s$ is connected, $t – s$ is disjoint from either $v$ or $w$. Suppose first that $t – s$ is disjoint from $v$. Then $t – s \subset w$, so $t \subset w \cup s$, and therefore the open subsets $w \cup s$ and $v – t$ of $u$ form a disconnection of $u$. I never would have thought of that! The other case is analogous. Let’s go back four pages to the Courier verbatim version of John’s proof and see if we can spot John’s nice argument... I didn’t see it.
Euclid was very creative and should be thought of as a founder of non-Euclidean geometry. He had many wonderful proofs which avoided the parallel axiom, including the triangle inequality, which we’d prove with the Pythagorean theorem. But none of Euclid’s proofs which involve angle addition are rigorous, including the triangle sum theorem. Euclid’s errors were fixed by Hilbert’s work. Angle addition problems are solved by betweenness axioms, e.g. Pasch’s axiom (a line intersecting a triangle side intersect a different side).

To see that Euclid had angle addition problems, consider alternate interior angles, which Euclid didn’t define. We’re suppose to note which angles are alternate interior by looking at our picture. That’s not how axiomatic proofs work.
Hilbert’s book *Foundations of Geometry*, first published in 1902, is not a good text to understand how to fix Euclid’s Elements. There are two good texts on Hilbert axiomatic geometry, *Euclidean and non-Euclidean Geometries* by Greenberg, and *Geometry, Euclid and Beyond* by Hartshorne. Greenberg’s book is quite rigorous, but it’s not primarily about Euclid. Hartshorne beautifully explains how Hilbert’s axioms fix Euclid, but his book contains many errors. A good proof assistant was therefore quite valuable to me, and Freek’s miz3 was quite easy to learn how to use. The only difficulty was understanding enough of the HOL Light type theory to write definitions.
Another reason for my Hilbert project is that Greenberg and Hartshorne use a stronger than necessary version of Hilbert’s axioms. A weaker version of axioms is found in Venema’s book *Foundations of Geometry*, and I believe his axioms come from a much later version of FoG. Venema’s book is quite rigorous, and it follows Moise’s book *Elementary Geometry from an Advanced Standpoint* which uses the real line and axioms for measuring angles and distances. Moise was I believe an influence on Greenberg, and he was the leader of the SMSG project to improve K-12 math education. Moise wrote (with Downs) a Geometry text with quite respectable Hilbert rigor, now unfortunately out of print.
I should mention the related axiomatic geometry project of Julien Narboux, who used Coq to formalize Tarski’s work. Tarski’s axioms are much weaker than Hilbert’s, and Tarski never wrote up a proof of his well-known result that Euclidean geometry is decidable. The reason for decidability has nothing to do with the weakness of Tarski’s axioms, but the fact that Tarski included a first-order version of the 2nd order Dedekind cuts Hilbert axiom. As Hartshorne explains, Hilbert proved that including the Dedekind cuts axioms implies that Euclidean 2-space is the unique model of the axioms. Tarski needed to prove a first order version of this. The only published source for Tarski’s work is an unrefereed book by Schwabhäuser (called SST, as it’s coauthed by Szmielew and Tarski), written in German and published by Ishi press, a publisher of Go books. We’re indebted to Michael Beeson for getting SST published. So formalizing SST is a great idea.
Hilbert’s axioms for plane geometry

We assume we have a set of points, which we call the Hilbert plane, together with a set of subsets called lines, open interval (A, B) for any distinct points A and B, and two binary relations ≡ on segments and angles which will be defined later. We write Collinear A B C if there is a line l containing the points A, B and C. First the incidence axioms:

I1 ⊢ ∀ A B. ¬(A = B) ⇒ (∃! l. Line l ∧ A ∈ l ∧ B ∈ l)
I2 ⊢ ∀ l. Line l ⇒ (∃ A B. A ∈ l ∧ B ∈ l ∧ ¬(A = B))
I3 ⊢ ∃ A B C. ¬(A = B) ∧ ¬(A = C) ∧ ¬(B = C) ∧ ¬Collinear A B C
B1’ ⊬ ∀ A B C. B ∈ open (A,C) ⊨ ¬(A = B) ∧ ¬(A = C) ∧ ¬(B = C) ∧ B ∈ open (C,A) ∧ Collinear A B C
B2’ ⊬ ∀ A B. ¬(A = B) ⇒ (∃ C. B ∈ open (A,C))
B3’ ⊬ ∀ A B C.
¬(A = B) ∧ ¬(A = C) ∧ ¬(B = C) ∧ Collinear A B C ⊨ (B ∈ open (A,C) ∨ C ∈ open (B,A) ∨ A ∈ open (C,B)) ∧ ¬(B ∈ open (A,C) ∧ C ∈ open (B,A)) ∧ ¬(B ∈ open (A,C) ∧ A ∈ open (C,B)) ∧ ¬(C ∈ open (B,A) ∧ A ∈ open (C,B))
B4’ ⊬ ∀ l A B C.
Line l ∧ ¬Collinear A B C ∧ A ∉ l ∧ B ∉ l ∧ C ∉ l ∧ (∃ X. X ∈ l ∧ X ∈ open (A,C)) ⊨ (∃ Y. Y ∈ l ∧ Y ∈ open (A,B)) ∨ (∃ Y. Y ∈ l ∧ Y ∈ open (B,C))
The segment congruence axioms

We define a segment to be a “closed interval”, and then define the predicate SEGMENT:

Segment.DEF ⊢ ∀ A B. seg A B = A, B ∪ open (A,B)
SEGMENT ⊢ ∀ s. Segment s ⇔ ∃ A B. s = seg A B ∧ ¬(A = B)

C2Reflexive ⊢ Segment s ⇒ s ≡ s
C2Symmetric ⊢ Segment s ∧ Segment t ∧ s ≡ t ⇒ t ≡ s
C2Transitive ⊢ Segment s ∧ Segment t ∧ Segment u ∧ s ≡ t ∧ t ≡ u ⇒ s ≡ u

C3 ⊢ ∀ A B C A’ B’ C’.
B ∈ open (A,C) ∧ B’ ∈ open (A’,C’) ∧ seg A B ≡ seg A’ B’ ∧ seg B C ≡ seg B’ C’
⇒ seg A C ≡ seg A’ C’
\[ \text{Ray\_DEF} \vdash \forall A\ B. \text{ray } A\ B = \{ X \mid \neg (A = B) \land \text{Collinear } A\ B\ X \land A \notin \text{open } (X,B) \} \]

\[ \text{RAY} \vdash \forall r. \text{Ray } r \iff \exists O\ A. \neg (O = A) \land r = \text{ray } O\ A \]

\[ \text{Angle\_DEF} \vdash \forall A\ O\ B. \angle A\ O\ B = \text{ray } O\ A \cup \text{ray } O\ B \]

\[ \text{ANGLE} \vdash \forall \alpha. \text{Angle } \alpha \iff \exists A\ O\ B. \alpha = \angle A\ O\ B \land \neg \text{Collinear } A\ O\ B \]

\[ \text{SegmentOrdering\_DEF} \vdash \forall t\ s. \ s < \_\text{seg } t \iff \text{Segment } s \land \exists C\ D\ X. t = \text{seg } C\ D \land X \in \text{open } (C,D) \land s \equiv \text{seg } C\ X \]

\[ \text{AngleOrdering\_DEF} \vdash \forall \beta\ \alpha. \ \alpha < \_\text{ang } \beta \iff \text{Angle } \alpha \land \exists A\ O\ B\ G. \neg \text{Collinear } A\ O\ B \land \beta = \angle A\ O\ B \land G \in \text{int} \_\text{angle } A\ O\ B \land \alpha \equiv \angle A\ O\ G \]

\[ \text{SameSide\_DEF} \vdash \forall l\ A\ B. \ A,B \text{ same}_\text{side } l \iff \text{Line } l \land \neg \exists X. X \in l \land X \in \text{open } (A,B) \]
The angle congruence axioms

C4 ⊢ ∀ α O A l Y.
   Angle α ∧ ¬(O = A) ∧ Line l ∧ O ∈ l ∧ A ∈ l ∧ Y ∉ l
   ⇒ ∃! r. Ray r ∧ ∃ B. ¬(O = B) ∧ r = ray O B ∧
   B ∉ l ∧ B,Y same_side l ∧ ∠ A O B ≡ α
C5Reflexive ⊢ Angle α ⇒ α ≡ α
C5Symmetric ⊢ Angle α ∧ Angle β ∧ α ≡ β ⇒ β ≡ α
C5Transitive ⊢ Angle α ∧ Angle β ∧ Angle γ ∧ α ≡ β ∧ β ≡ γ
     ⇒ α ≡ γ
C6 ⊢ ∀ A B C A’ B’ C’.
   ¬Collinear A B C ∧ ¬Collinear A’ B’ C’
   ∧ seg B A ≡ seg B’ A’ ∧ seg B C ≡ seg B’ C’ ∧
   ∠ A B C ≡ ∠ A’ B’ C’ ⇒ ∠ B C A ≡ ∠ B’ C’ A’
simple lemmas needed for Euclid Prop I.27

NonCollinearRaa \vdash \forall A \ B \ C \ I.

\neg (A = C) \ \land \ \text{Line} \ I \ \land \ A \ \in \ I \ \land \ C \ \in \ I \ \land \ B \ \notin \ I \ \Rightarrow \ \neg \text{Collinear} \ A \ B \ C

CollinearSymmetry \vdash \forall A \ B \ C. \ \text{Collinear} \ A \ B \ C

\Rightarrow \ \text{Collinear} \ A \ C \ B \ \land \ \text{Collinear} \ B \ A \ C \ \land

\text{Collinear} \ B \ C \ A \ \land \ \text{Collinear} \ C \ A \ B \ \land \ \text{Collinear} \ C \ B \ A

Collinear\_DEF

\vdash \forall A \ B \ C. \ \text{Collinear} \ A \ B \ C \ \iff \ \exists I. \ \text{Line} \ I \ \land \ A \ \in \ I \ \land \ B \ \in \ I \ \land \ C \ \in \ I

NoncollinearityExtendsToLine

\vdash \forall A \ O \ B \ X. \ \neg \text{Collinear} \ A \ O \ B \ \land \ \text{Collinear} \ O \ B \ X \ \land \ \neg (X = O)

\Rightarrow \ \neg \text{Collinear} \ A \ O \ X

SameSideSymmetric

\vdash \forall I \ A \ B. \ \text{Line} \ I \ \land \ A \ \notin \ I \ \land \ B \ \notin \ I \ \land \ A,B \ \text{same\_side} \ I

\Rightarrow \ B,A \ \text{same\_side} \ I
some harder lemmas

SameSideTransitive
⊢ ∀ I A B C. Line I ∧ A ∉ I ∧ B ∉ I ∧ C ∉ I ∧ A,B same_side I ∧ B,C same_side I
⇒ A,C same_side I

RaySameSide
⊢ ∀ I O A P. Line I ∧ O ∈ I ∧ A ∉ I ∧ P ∈ ray O A - O
⇒ P ∉ I ∧ P,A same_side I

RayWellDefined
⊢ ∀ O P Q. ¬(Q = O) ∧ P ∈ ray O Q - O ⇒ ray O P = ray O Q

AtMost2Sides
⊢ ∀ A B C I. Line I ∧ A ∉ I ∧ B ∉ I ∧ C ∉ I
⇒ A,B same_side I ∨ A,C same_side I ∨ B,C same_side I

AngleTrichotomy1
⊢ ∀ α β. α < ‾ang β ⇒ ¬(α ≡ β)
Euclid’s big gun: the exterior angle theorem

\[ \forall A B C D. \neg \text{Collinear } A B C \land C \in \text{open } (B,D) \Rightarrow \angle B A C < \angle D C A \]

Hartshorne proves this nicely. Euclid uses this to avoid using the parallel axiom. We need the definition of parallel:

\[ \text{PARALLEL} \]
\[ \forall I K. I \parallel K \iff \text{Line } I \land \text{Line } K \land I \cap K = \emptyset \]

We won’t use it, but here’s the parallel axiom:

\[ \text{P} \vdash \forall P I. \text{Line } I \land P \notin I \Rightarrow \exists! M. \text{Line } M \land P \in M \land M \parallel I \]
The Alternate Interior Angles Theorem

let AlternateInteriorAngles = theorem ‘;
∀A B C E l m t. Line l ∧ A ∈ l ∧ E ∈ l ⇒
Line m ∧ B ∈ m ∧ C ∈ m ⇒ Line t ∧ A ∈ t ∧ B ∈ t ⇒
¬(A = E) ∧ ¬(B = C) ∧ ¬(A = B) ∧ E ∉ t ∧ C ∉ t ⇒
¬(C,E same_side t) ⇒ ∠ E A B ≡ ∠ C B A
⇒ l || m

proof
intro_TAC ∀A B C E l m t, l_line, m_line, t_line, Distinct,
Cnsim_tE, AltIntAngCong;
¬Collinear E A B ∧ ¬Collinear C B A [EABncol] by fol t_line
Distinct NonCollinearRaa CollinearSymmetry;
B ∉ l ∧ A ∉ m [notAmBl] by fol l_line m_line Collinear_DEF - ∉;
assume ¬(l || m) [Con] by fol;
¬(l ∩ m = ∅) [] by fol - l_line m_line PARALLEL;
consider G such that \( G \in l \land G \in m \) [Glm] by fol - MEMBER_NOT_EMPTY_IN_INTER;

\( \neg (G = A) \land \neg (G = B) \land \text{Collinear } B \ G \ C \land \text{Collinear } B \ C \ G \land \text{Collinear } A \ E \ G \land \text{Collinear } A \ G \ E \) [GnotAB] by fol - notAmBl \( \notin m \_line l \_line \text{Collinear} \_DEF; \)

\( \neg \text{Collinear } A \ G \ B \land \neg \text{Collinear } B \ G \ A \land G \notin t \) [AGBncol] by fol EABncol CollinearSymmetry - NoncollinearityExtendsToLine t\_line {Collinear} _DEF \( \notin; \)

\( \neg (E,C \text{ same } \text{side } t) \) [Ensim\_tC] by fol t\_line - Distinct Cnsim\_tE SameSideSymmetric;

\( E \in l\ \text{DIFF } A \land G \in l\ \text{DIFF } A \) [] by fol l\_line Glm Distinct GnotAB IN\_DIFF IN\_SING;

\( \neg (G,E \text{ same } \text{side } t) \) [] proof assume G,E same \_side t [Gsim\_tE] by fol;

\( A \notin \text{Open } (G, \ E) \) [notGAE] by fol t\_line - SameSide\_DEF \( \notin; \)
\[ G \in \text{ray} \ A \ E \ \text{DIFF} \ A \ [\] \text{by fol} \ \text{Distinct} \ G\not\in AB \not\in GAE \ \text{IN}_{-}\text{Ray} \]
\[ G\not\in AB \ \text{IN}_{-}\text{DIFF} \ \text{IN}_{-}\text{SING}; \]
\[ \text{ray} \ A \ G = \text{ray} \ A \ E \ [rAGrAE] \text{by fol} \ \text{Distinct} \ -\ \text{RayWellDefined}; \]
\[ \neg (C, G \text{ same side } t) [\text{Cnsim}_{-}tG] \text{by fol } t\text{-line} \ AGBncol \ \text{Distinct} \]
\[ G\sim_{-}tE \ \text{Cnsim}_{-}tE \ \text{SameSideTransitive}; \]
\[ C \notin \text{ray} \ B \ G [\text{notCrBG}] \text{by fol} - \text{IN}_{-}\text{Ray} \ \text{Distinct} \ t\text{-line} \ AGBncol \]
\[ \text{RaySameSide} \ \text{Cnsim}_{-}tG \ \text{IN}_{-}\text{DIFF} \ \text{IN}_{-}\text{SING} \notin; \]
\[ B \in \text{Open} \ (C, G) [\] \text{by fol} - \ G\not\in AB \notin \text{IN}_{-}\text{Ray}; \]
\[ \angle G \ A \ B < \ _{\text{ang}} \ C \ B \ A [\] \text{by fol} \ AGBncol \ \text{notCrBG} - B1’ \]
\[ \text{EuclidPropositionI}_{-}16; \]
\[ \angle E \ A \ B < \ _{\text{ang}} \ C \ B \ A [\] \text{by fol} - rAGrAE \ \text{Angle}_{-}\text{DEF}; \]
\[ \text{fol} \ \text{EABncol} \ \text{ANGLE} \ \text{AltIntAngCong} - \ \text{AngleTrichotomy1}; \]
\[ \text{qed}; \]
\[ G, C \text{ same side } t [G\sim_{-}tC] \text{by fol } t\text{-line} \ AGBncol \ \text{Distinct} - \]
\[ \text{Cnsim}_{-}tE \ \text{AtMost2Sides}; \]
\[ B \notin \text{Open} \ (G, C) [\text{notGBC}] \text{by fol } t\text{-line} - \text{SameSide}_{-}\text{DEF} \notin; \]

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G ∈ ray B C DIFF B [] by fol Distinct GnotAB notGBC IN_Ray
GnotAB IN_DIFF IN_SING;
ray B G = ray B C [rBGrBC] by fol Distinct - RayWellDefined;
∠ C B A ≡ ∠ E A B [flipAltIntAngCong] by fol EABncol ANGLE
AltIntAngCong C5Symmetric;
¬(E,G same_side t) [Ensim_tG] by fol t_line AGBncol Distinct
Gsim_tC Ensim_tC SameSideTransitive;
E ∉ ray A G [notErAG] by fol - IN_Ray Distinct t_line AGBncol
RaySameSide Ensim_tG IN_DIFF IN_SING ∉;
A ∈ Open (E, G) [] by fol - GnotAB ∉ IN_Ray;
∠ G B A < _ang∠ E A B [] by fol AGBncol notErAG - B1'
EuclidPropositionI_16;
∠ C B A < _ang∠ E A B [] by fol - rBGrBC Angle_DEF;
fol EABncol ANGLE flipAltIntAngCong - AngleTrichotomy1;
qed;
;;;;