Math 226: Sequences and Series
Northwestern University, Lecture Notes

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These are notes which provide a basic summary of each lecture for Math 226, "Sequences and Series", taught by the author at Northwestern University. The book used as a reference is the 14th edition of *Thomas' Calculus* by Hass, Heil, and Weir. Watch out for typos! Comments and suggestions are welcome.

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Lecture 1: Sequences

The goal of this course is to understand *series* and their use in mathematics. A *series* is an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

obtained by adding together infinitely many quantities. (The $\cdots$ are meant to say "keep going without end.") It is surprising that a definite meaning can be given to such an "infinite sum", since it is not possible to actually sit down and perform the required addition by hand—it would take an infinite amount of time! Nonetheless, we will see how to make sense of this, and why equalities such as

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2$$

are true. (On the left we are adding reciprocals of positive integers with alternating signs.) This equality says that if we *could* actually perform the infinite summation on the left, we would get the value of $\ln 2$, but again the point is that we can determine this without having to actually perform this infinite summation.

Series show up in various contexts, and in particular are heavily used in techniques developed to *approximate* functions. For instance, the values of the function $f(x) = e^x$ turn out to also be given by the following infinite series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

which can be viewed as an "infinite" polynomial. As a consequence, the polynomial $1 + x$ provides an approximation to the value of e^x, the polynomial $1 + x + \frac{x^2}{2!}$ provides a better approximation, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ an even better approximation, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ even better, and so on. This final approximation

To see this in action, note that the value of e^1 is approximately

$$e^1 \approx 2.7182818285$$

which is as many decimal places as my computer whould show me. Plugging $x = 1$ into the polynomials above gives the values:

$$1+1=2 \qquad 1+1+\frac{1^2}{2!}=2.5 \qquad 1+1+\frac{1^2}{2!}+\frac{1^3}{3!}\approx 2.667 \qquad 1+1+\frac{1^2}{2!}+\frac{1^3}{3!}+\frac{1^4}{4!}\approx 2.708$$

which are indeed getting closer and closer to the actual value of e^1. This final approximation is called a *fourth-order* approximation since it came from a polynomial of degree 4, and is the first approximation which agrees with the actual value of e^1 to the first decimal digit. A question we can ask is: could we determine beforehand that we would need to use the fourth-order approximation in order to obtain a value which is accurate to first decimal digit? What if we wanted an approximation which was accurate to two decimals, or more? These are questions we will explore as well.

But before all this, we must understand the notion of a series better. Our first overall goal is to understand when a series expressed as an infinite sum gives a well-defined value, since it is highly non-obvious that adding together infinitely many numbers can still give a finite value as a result.

Sequences. Before talking about series, we must first talk about *sequences*, which are the things from which series are built. A sequence is nothing but an infinite list of numbers:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots.$$

(The connection between sequences and series is that a series is obtained by adding together the terms of a sequence.) To say that a sequence *converges* to a number L—which we call the *limit* of the sequence—is to say that the terms of the sequence get closer and closer to L the further along the sequence we go. Written in terms of limits, a sequence whose n-th term is denoted by a_n converges to a number L when

$$\lim_{n\to\infty} a_n = L.$$

To say that a sequence *diverges* just means that it does not converge.

Example 1. Consider the sequence

$$a_n = 2 + \frac{(-1)^n}{n^2}.$$

To be clear, this notation refers to the sequence whose n-th term is the given a_n. So, the first term of this sequence is teh value a_1 (when $n = 1$), the second term is a_2 (when $n = 2$), and so on. The first few terms of this sequence are thus

$$2 + \frac{-1}{1^2},\ 2 + \frac{1}{2^2},\ 2 + \frac{-1}{3^2},\ 2 + \frac{1}{4^2},\ \ldots.$$

As n goes to infinity, the $\frac{(-1)^n}{n^2}$ gets closer and closer to zero since the numerator bounces back forth between -1 and 1 while the denominator gets larger and larger. Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2 + \frac{(-1)^n}{n^2}\right) = 2 + 0 = 2,$$

so this sequence converges to 2.

Example 2. The sequence

$$b_n = n^2$$

diverges. To be clear, the first few terms of this sequence are

$$1,\ 4,\ 9,\ 16,\ 25,\ \ldots,$$

and the point is that these terms get larger and larger, and thus grow without bound. In particular then, these terms cannot be approaching any one definite value. (We could say that this sequence *diverges to infinity*, meaning that its terms keep getting larger and larger and larger.)

Example 3. The sequence

$$c_n = 1 + (-1)^n$$

also diverges, but for a different reason than that in the previous example. Here the sequence looks like:

$$0,\ 2,\ 0,\ 2,\ 0,\ 2,\ \ldots$$

so this sequence consists of alternating 0's and 2's. Here it's not that these terms get larger and larger, but rather that the alternating behavior prevents them from approaching any one specific value. (So, this sequence diverges, but would *not* say that it "diverges to infinity".)

Example 4. Consider the sequence

$$d_n = \frac{3n^2 - 3n + 1}{2n^2 + 4n - 1}$$

whose first few terms are:

$$\frac{1}{5}, \frac{7}{15}, \frac{19}{29}, \ldots$$

In this case it is not clear yet whether the sequence should converge or diverge. To get some intuition, the key observation is that the n^2 terms in the numerator and denominator "dominate" all the other terms in the sense that they should determine the overall behavior because they end up being the largest terms involved as n gets larger and larger. In other words, this sequence should in a sense behave similarly to the constant sequence

$$\frac{3n^2}{2n^2} = \frac{3}{2}$$

which converges to $\frac{3}{2}$. Hence the sequence a_n converges to $\frac{3}{2}$. So, we might guess that our sequence will converge to $\frac{3}{2}$ as well.

In order to make this guess precise we recall some things about manipulating limits from previous calculus courses. The point is that we cannot simply take the limit as $n \to \infty$ of the numerator and denominator just yet because neither of these limits actually exist: both the numerator and denominator individually describe sequence which diverge to infinity. Here we can rewrite the given sequence by dividing the entire numerator and denominator each by n^2. We get:

$$\lim_{n\to\infty} \frac{3n^2 - 3n + 1}{2n^2 + 4n - 1} = \lim_{n\to\infty} \frac{3 - \frac{3}{n} + \frac{1}{n^2}}{2 + \frac{4}{n} - \frac{1}{n^2}}.$$

The $\frac{3}{n}$, $\frac{1}{n^2}$, $\frac{4}{n}$, and $\frac{1}{n^2}$ terms go to 0 as n goes to infinity, so we get

$$\lim_{n\to\infty} \frac{3 - \frac{3}{n}}{2 + \frac{4}{n} - \frac{1}{n^2}} = \frac{3 - 0}{2 + 0 - 0} = \frac{3}{2}$$

as the value of the limit. In this final step, we are able to indeed take the limit of the numerator and denominator separately since each of these limits actually exist.

A similar technique works for any sequence defined by taking a fraction of polynomials. In general, when the highest power of n in the numerator is greater than that in the denominator, the sequence will diverge; when the highest power of n in the numerator is smaller than that in the denominator, the sequence will converge to 0; and when these highest powers of n are the same, the sequence will converge to the fraction obtained by taking the coefficients of these highest powers. However, you SHOULD work out such limits carefully using the same technique as above where we divide numerator and denominator by a power of n to simplify the given expression.

Example 5. Finally we consider the sequence

$$a_n = \frac{2 + \sin n}{n}.$$

Intuitively, the $\sin n$ term can only have values between -1 and 1, so the fraction

$$\frac{\sin n}{n}$$

should have limit 0 as $n \to \infty$ because we have a numerator which is constrained between -1 and 1 with a denominator that keeps getting larger and larger. If we had a sequence defined by dividing a *constant* by n, like $\frac{3}{n}$, this would be enough to say that the limit was zero, but here we should be more careful: $\sin n$ is not constant, and we cannot make this jump directly.

To make this more precise, we argue as follows. Since $\sin n$ is always between -1 and 1, we have the inequalities:

$$\frac{2+(-1)}{n} \leq \frac{2+\sin n}{n} \leq \frac{2+1}{n},$$

or in other words

$$\frac{1}{n} \leq \frac{2+\sin n}{n} \leq \frac{3}{n}.$$

The sequence on the left converges to 0, as does the sequence on the right. So, the sequence we actually care about is "sandwiched" between terms which are themselves getting closer and closer to zero, but this then actually implies that the terms in the middle are *also* getting closer and closer to zero! so, we conclude that

$$\lim_{n\to\infty} \frac{2+\sin n}{n} = 0$$

as expected by our intuition.

Sandwich Theorem. The technique used in the final example is important enough that we give it a special name: the *Sandwich Theorem.* To be clear, this says that if we have three sequences $a_n, b_n c_n$ related by inequalities

$$a_n \leq b_n \leq c_n,$$

then if the two "outer" sequences a_n and c_n both converge to the same number L, so does the sequence in the middle: if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$. Again, the intuition is that terms b_n are "sandwiched" between terms that are each getting closer and closer to L, so the terms b_n must be getting closer to L as well. Note that this fact also goes by the name "Squeeze Theorem", which you might come across in other references.

Lecture 2: More on Sequences

Warm-Up 1. We determine whether or not the sequence defined by

$$a_n = \frac{2\cdot 3^{n+1}}{5^n} + \frac{(3n-1)!}{(3n+1)!}$$

converges. First, we can rewrite the first part of the sum as

$$\frac{2\cdot 3^{n+1}}{5^n} = \frac{2\cdot 3\cdot 3^n}{5^n} = 6\left(\frac{3}{5}\right)^n.$$

As n gets larger and larger, $(\frac{3}{5})^n$ gets smaller and smaller:

$$\frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \ldots$$

since the denominator is getting larger at a much faster rate than the numerator. Thus $(\frac{3}{5})^n$ converges to 0, so

$$\lim_{n\to\infty} \frac{2\cdot 3^{n+1}}{5^n} = \lim_{n\to\infty} 6\left(\frac{3}{5}\right)^n = 0.$$

(In general, if r is any number strictly between -1 and 1, then the sequence r^n converges to 0 as well. This will be an important fact when we study what are called *geometric series*.)

Now, the second part of the sum in the definition of a_n can be written as:

$$\frac{(3n-1)!}{(3n+1)!} = \frac{(3n-1)!}{(3n+1)(3n)(3n-1)!} = \frac{1}{(3n+1)(3n)}.$$

To be clear, $(3n+1)$ is the product of $3n+1$ and all positive integers before it:

$$(3n+1)! = (3n+1)(3n)(3n-1)(3n-2)(3n-3)\cdots 3\cdot 2\cdot 1,$$

and the portion of this which looks like $(3n-1)(3n-2)\cdots 3\cdot 2\cdot 1$ is precisely $(3n-1)!$. So

$$\lim_{n\to\infty} \frac{(3n-1)!}{(3n+1)!} = \lim_{n\to\infty} \frac{1}{(3n-1)(3n)} = 0$$

since the numerator is constant and the denominator gets larger and larger.

Since both parts of the sum in the definition of a_n individually converge, we can say that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2\cdot 3^{n+1}}{5^n} + \frac{(3n-1)!}{(3n+1)!}\right) = \lim_{n\to\infty}\frac{2\cdot 3^{n+1}}{5^n} + \lim_{n\to\infty}\frac{(3n-1)!}{(3n+1)!} = 0+0=0,$$

so our given sequence a_n converges to 0.

To highlight a fact we used: if $\lim_{n\to\infty} b_n$ and $\lim_{n\to\infty} c_n$ each exist individually, then

$$\lim_{n\to\infty}(b_n + c_n) = \lim_{n\to\infty} b_n + \lim_{n\to\infty} c_n.$$

But take note that we would not be able to "split up" a sum in this way if either limit of b_n or c_n did not exist on its own.

Warm-Up 2. Next we consider the sequence

$$b_n = \frac{2n-3}{10\sqrt{n}+4}.$$

Here is some intuition: the numerator behaves more and more like the $2n$ term alone (in other words, the $2n$ term "dominates" the others in the numerator) and the denominator behaves more and more like $10\sqrt{n}$ (i.e. $10\sqrt{n}$ "dominates" the denominator), so this sequence should behave in way similar to the sequence

$$\frac{2n}{10\sqrt{n}} = \frac{\sqrt{n}}{5},$$

which diverges to infinity. So we can make a good guess that our given sequence should diverge to infinity as well.

To be precise, we can rewrite our given sequence by dividing everything by the largest power of n which appears, so in this case divide numerator and denominator by n:

$$\frac{2n-3}{10\sqrt{n}+4} = \frac{2-\frac{3}{n}}{\frac{10}{\sqrt{n}}+\frac{4}{n}}.$$

As $n\to\infty$, the fractions $\frac{3}{n}, \frac{10}{\sqrt{n}}$, and $\frac{4}{n}$ all converge to 0, so the numerator above converges to 2 while the denominator converges to 0. Thus the entire fraction will diverge to infinity since it has a numerator getting closer and closer to a nonzero constant, being divided by numbers which are getting smaller and smaller, which causes the fraction itself to get larger and larger.

As an alternate approach, we also divide the numerator and denominator of our original sequence fraction by $\sqrt{n}$ to rewrite it as:

$$\frac{2n-3}{10\sqrt{n}+4} = \frac{2\sqrt{n} - \frac{3}{\sqrt{n}}}{10 + \frac{4}{\sqrt{n}}}.$$

As n gets larger and larger, $\frac{3}{\sqrt{n}}$ and $\frac{4}{\sqrt{n}}$ both converge to 0. Hence the denominator converges to 10 but the numerator diverges to infinity since the $2\sqrt{n}$ terms grows without bound. Thus the sequence $\frac{2n-3}{10\sqrt{n}+4}$ diverges to infinity as well.

Warm-Up 3. Finally we look at the sequence

$$b_n = \frac{n!}{n^n}.$$

Again, both the numerator and denominator are going to infinity, but the fact that the denominator goes to infinity more quickly than the numerator suggests that b_n might converge to zero. However, to justify this requires some care.

Note that b_n is always bigger than or equal to zero. If we write out what the numerator and denominator look like, we get:

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdot n \cdots n}.$$

To be clear, we wrote out $n!$ as "1 times 2 times 3 times $\cdots$ all the way up to n", and we write out n^n has n times itself n times. Note that each term in the numerator is smaller than or equal to the corresponding term in the denominator below it. In particular, in:

$$0 \leq \frac{n!}{n^n} = \frac{1}{n}\left[\frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n}\right],$$

the fraction in brackets is less than 1 since the entire numerator is smaller than the entire denominator. This says that the entire term on the right is less than $\frac{1}{n} \cdot 1 = \frac{1}{n}$, so

$$0 \leq \frac{n!}{n^n} = \frac{1}{n}\left[\frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n}\right] \leq \frac{1}{n}.$$

The terms 0 on the left converge to 0 and the term $\frac{1}{n}$ on the right converges to 0, so the Sandwich Theorem we saw last time says that the term $b_n = \frac{n!}{n^n}$ in the middle converges to 0 as well! So we get

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

One point to make here is that it is not at all obvious that we should be looking to compare our given sequence $\frac{n!}{n^n}$ with the simpler sequence $\frac{1}{n}$, nor how the comparison should actually work out. This is the type of things which comes with practice and having seen enough examples; not that you have seen this particular comparison used in this example, you can be on the lookout for similar things which might work in related examples.

Using continuous functions. Building off the last example, suppose we now look at the sequence

$$a_n = \cos\left(\frac{n!}{n^n}\right).$$

We know that the expression $\frac{n!}{n^n}$ at which cosine is being evaluated itself converges to 0, and so we might be tempted to say that cosine of this expression should converge to $\cos 0$:

$$\lim_{n\to\infty} \cos\left(\frac{n!}{n^n}\right) = \cos 0 = 1 \text{since} \lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

This is true! But, we should be clear about the reason why.

The point is that this works because the function $f(x) = \cos x$ is a *continuous* function, of the type you would have seen in a previous calculus course. This is the property which guarantees that we can interchange a limit operation and function in the following way:

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

Thus, since the cosine function is continuous, we do indeed have

$$\lim_{n\to\infty} \cos\left(\frac{n!}{n^n}\right) = \cos\left(\lim_{n\to\infty} \frac{n!}{n^n}\right) = \cos 0 = 1.$$

Similarly, since the function $g(x) = e^x$ is continuous, we could also say that

$$\lim_{n\to\infty} e^{\frac{n!}{n^n}} = e^{\lim_{n\to\infty} \frac{n!}{n^n}} = e^0 = 1.$$

This is useful in computing limits of sequences in that it allows us to focus on the specific part of the limit which actually matters, say the $\frac{n!}{n^n}$ terms in the two examples above.

For an example where this doesn't work if our function is not continuous, consider the function $h(x)$ defined by

$$h(x) = \begin{cases} \cos x & \text{if } x > 0 \\ 10 & \text{if } x \leq 0. \end{cases}$$

Here it is NOT true that

$$\lim_{n\to\infty} h\left(\frac{n!}{n^n}\right) = h\left(\lim_{n\to\infty} \frac{n!}{n^n}\right)$$

since the left-hand side is $\lim_{n\to\infty} \cos\left(\frac{n!}{n^n}\right) = \cos 0 = 1$ but the right-hand side is $h(0) = 10$. The "jump" which this function has at $x = 0$ prevents this limit property from working out.

Example with L'Hopital's rule. Consider the sequence

$$a_n = n^2 e^{-n}, \text{ which can be written as } a_n = \frac{n^2}{e^n}.$$

Note that here both the numerator and denominator are getting larger and larger (so going to ∞) as n goes to infinity. However, the denominator goes to ∞ much faster than the numerator, which suggests the sequence should converge to 0. To make this precise we need to use L'Hopital's rule. But, L'Hopital's rule is a technique used for *functions* not sequences, so we should be clear about how it is exactly that we're applying it.

Consider the function

$$f(x) = \frac{x^2}{e^x}.$$

The sequence we're looking at is the sequence $a_n = f(n)$ obtained by plugging in whole numbers n into f, which suggests that it should be possible to determine the behavior of our sequence by looking at the behavior of this function instead. (But, to be sure, this function and our sequence are *technically* different things, although they are related.) We want to determine

$$\lim_{x\to\infty} \frac{x^2}{e^x}.$$

Since the numerator and denominator both go to ∞, L'Hopital's rule applies and says that we can attempt to compute the limit by taking the derivative of the numerator and denominator:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

In this new fraction, the numerator and denominator each again go to ∞, so we can apply L'Hopital's rule to get:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x}.$$

This final limit is zero since the numerator has limit 2 and the denominator ∞, so the fraction goes to 0. Hence

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$

Since the original sequence is obtained by plugging in whole numbers n into $f(x)$, we also get

$$\lim_{n\to\infty} n^2 e^{-n} = 0,$$

so a_n converges to 0. The point is that the values of our sequence are *among* the values of the function $f(x)$ above, so if the values of this more general function approach some value, so too should the values of our specific sequence.

Final example. Finally, we determine whether the sequence $b_n = (1 + \frac{1}{n})^n$ converges. Now, it does not make sense to say that since $\frac{1}{n}$ converges to 0 and 1 raised to any power is still 1, we should have

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} 1^n = 1.$$

This type of reasoning is NOT valid, since there is no limit property which will allow us to take the limit of only a certain part of our expression (such as the $\frac{1}{n}$ term) while leaving any other dependence on n as is. In other words, we must consider the entire expression $(1 + \frac{1}{n})^n$ as a whole to determine the behavior, and not simply the exponent n alone or the $1 + \frac{1}{n}$ being exponentiated. Another comment is that $(1 + \frac{1}{n})^n$ is the product of a bunch of factors, but the *number* of factors changes as n changes:

$$\left(1 + \frac{1}{1}\right), \ \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2}\right), \ \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{3}\right), \ \ldots.$$

Standard limit properties like $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$ only work when the number of factors considered is not change itself.

So, we need something new here. The key point is that we rewrite the given sequence expression in an alternate way, using the fact that

$$x = e^{\ln x}$$

for any positive number x. In our case, this means we can rewrite $(1 + \frac{1}{n})^n$ as

$$\left(1 + \frac{1}{n}\right)^n = e^{\ln\left(1 + \frac{1}{n}\right)^n}.$$

But now, the natural log term in the exponent can be further written as

$$e^{\ln\left(1+\frac{1}{n}\right)^n} = e^{n\ln\left(1+\frac{1}{n}\right)}.$$

Thus we were able to rewrite our given sequence expression $(1+\frac{1}{n})^n$ in a way which no longer as something raised to the n-th power. Moreover, since e^x is continuous, we can go one step further and say:

$$\lim_{n\to\infty} e^{n\ln(1+\frac{1}{n})} = e^{\lim_{n\to\infty} n\ln(1+\frac{1}{n})}$$

so that in the end this all comes down to finding the limit of the sequence $n\ln(1+\frac{1}{n})$. The overarching idea here, as we have seen in other examples, is to find a way to rewrite our given sequence expression in a way which makes other limit properties applicable.

To compute the limit of the sequence $n\ln(1+\frac{1}{n})$, we consider the function $f(x) = x\ln(1+\frac{1}{x})$. Rewriting this as

$$f(x) = \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$$

and taking a limit as $x \to \infty$ gives a numerator and denominator which both approach 0. Thus L'Hopital's rule is applicable, and we get

$$\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

Since $\lim_{x\to\infty} f(x) = 1$, we also then have $\lim_{n\to\infty} f(n) = \lim_{nto\infty} n\ln(1+\frac{1}{n}) = 1$, so

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(1+\frac{1}{n})} = e^{\lim_{n\to\infty} n\ln(1+\frac{1}{n})} = e^1 = e.$$

Thus the sequence $b_n = (1+\frac{1}{n})^n$ converges to e. The idea used in this example, rewriting an expression using properties of e and ln as

$$\text{expression} = e^{\ln(\text{expression})},$$

is especially useful for sequences which involve n in both the exponent and thing being exponentiated. Such examples will often involve L'Hopital's rule as well.

Lecture 3: Newton's Method

Warm-Up 1. We show that the sequence

$$a_n = \sqrt[n]{n} = n^{1/n}$$

converges and determine its limit. As in the final example from last time, it is not the case that we can argue that since the n being exponentiated diverges the entire sequence will diverge, nor can we make immediate use of the fact that the exponent $\frac{1}{n}$ converges to 0; we must analyze the behavior of the entire expression as is. We can rewrite the given sequence expression as

$$a_n = n^{1/n} = e^{\ln(n^{1/n})} = e^{\frac{\ln n}{n}},$$

and hence will use the fact that the function $f(x) = e^x$ is continuous so that we can focus on computing the limit of the exponent first.

The function $g(x) = \frac{\ln x}{x}$ is one to which L'Hopital's rule is applicable since the numerator and denominator each go to ∞ as $x \to \infty$, so we get:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Thus, the sequence $g(n) = \frac{\ln n}{n}$ converges to 0 as well, so since $f(x) = e^x$ is continuous we have:

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1.$$

Warm-Up 2. Here is a new type of example. Define a sequence $x_1, x_2, x_3, x_4, \ldots$ by setting $x_1 = \sqrt{2}$ and then recursively defining each other term in terms of the previous one via:

$$x_{n+1} = \sqrt{2 + x_n} \text{ for } n \geq 1.$$

This says that each term in our sequence beyond x_1 is definee to be the square root of 2 plus the previous term. So for instance, when $n = 1$ we get

$$x_2 = \sqrt{2 + x_1} = \sqrt{2 + \sqrt{2}}.$$

When $n = 2$ we get

$$x_3 = \sqrt{2 + x_2} = \sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

Then the x_4 term is the square root of 2 plus x_3, which looks like:

$$x_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}},$$

and so on. Thus our sequence looks like

$$\sqrt{2},\ \sqrt{2 + \sqrt{2}},\ \sqrt{2 + \sqrt{2 + \sqrt{2}}},\ \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}},\ \ldots$$

where we keep getting "nested" expressions involving more and more $\sqrt{2}$'s.

The goal of this problem is to find the number to which this sequence converges. First we say something about why this sequence actually does converge. The first key observation is that this sequence is *increasing*, which means that each term is larger than the one which came before:

$$\sqrt{2} < \sqrt{2 + \sqrt{2}} < \sqrt{2 + \sqrt{2 + \sqrt{2}}} < \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} < \ldots$$

This happens because at each step we actually take the square root of a *larger* expression than before, which results in a larger term still. The second key observation is that all terms in this sequence are smaller than 2, so that this sequence is *bounded* meaning that its terms are constrained to lie within some interval of finite length; in this case, all terms are between $\sqrt{2}$ and 2. This happens because the expression of which we are taking the square root at each step is smaller than 4, so that the resulting square root is smaller than 2. (We are only giving some intuition for why this sequence is increasing and bounded, and actually justifying this carefully would take us outside the scope of this course, so we skip the precise justification here, but ask in office hours if you interested in learning more.)

The upshot is that it is a fact that any sequence which is increasing and bounded must converge! (Also, any sequence which is *decreasing* and bounded converges. We can summarize both of this cases in one statement by saying that any sequence which is *monotone* and bounded converges, where "monotone" refers to a sequence which is either increasing or decreasing.) Intuitively, if the terms of a sequence are getting larger at each step but never larger than a fixed value (2 in this case), then they should indeed "clump" towards some definite number, which will be its limit.

So, we know that this sequence x_n does converge, say to some to-be-determined value L. To figure out what L actually is, we go back to the recursive definition of our sequence:

$$x_{n+1} = \sqrt{2 + x_n}.$$

Let us the take the limit now of both sides as $n \to \infty$. On the right, we have

$$\lim_{n\to\infty} (2 + x_n) = 2 + L,$$

so since the square root function is continuous we get

$$\lim_{n\to\infty} \sqrt{2 + x_n} = \sqrt{\lim_{n\to\infty} (2 + x_n)} = \sqrt{2 + L}.$$

Now, what about the limit of the left side $\lim_{n\to\infty} x_{n+1}$? Let us write out some terms of this sequence. The first term when $n = 1$ is $x_{1+!} = x_2$, the second term when $n = 2$ is $x_{2+1} = x_3$, the third term is x_4, and so on:

$$x_2,\ x_3,\ x_4,\ x_5,\ \ldots$$

The point is that this is *almost* the same sequence as the original x_n:

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ \ldots$$

only with the first term x_1 missing! That is, x_{n+1} is the same sequence as x_n only with the terms "shifted" over to the left by one spot and with the x_1 term dropped. But, the behavior as $n \to \infty$ for this new sequence will be the same as that for the original sequence, simply because as we go further and further along in x_{n+1} we are getting the values as when we go further along in x_n. So, the shifted sequence x_{n+1} will *also* converge to L:

$$\lim_{n\to\infty} x_{n+1} = L.$$

Thus, the limit of the left side of

$$x_{n+1} = \sqrt{2 + x_n}$$

is L and the limit of the right side is $\sqrt{2 + L}$. But the left and right side sequences above are meant to be equal after all, so the resulting limits must be the same! Thus the to-be-determined value of L must satisfy

$$L = \sqrt{2 + L}$$

and we can now use this equality to actually figure out what L is. Solving for L (and recalling that L must be positive since the right $\sqrt{2 + L}$ is a positive number) gives $L = 2$, so we conclude that the sequence

$$\sqrt{2},\ \sqrt{2 + \sqrt{2}},\ \sqrt{2 + \sqrt{2 + \sqrt{2}}},\ \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}},\ \ldots$$

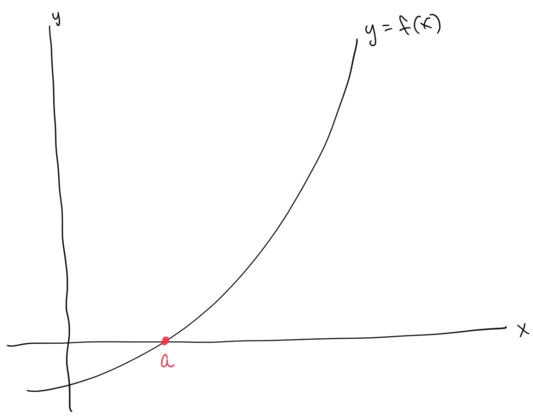
of nested $\sqrt{2}$ expressions will converge to 2. Surprising, no?

Newton's Method. Recursive sequences like the one in the second Warm-Up show up often in practice. As our final topic before moving on to discussing series we give one such application, called *Newton's Method*. The setup is some equation

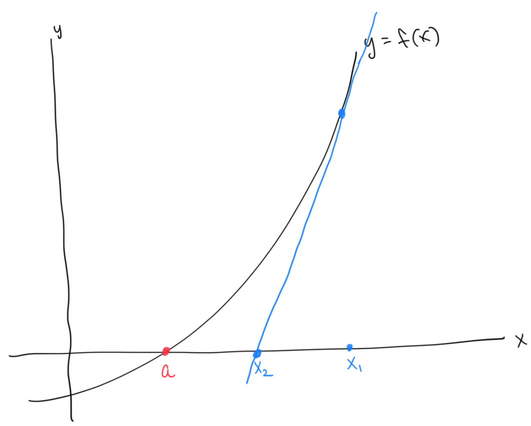
$$f(x) = 0$$

we want to solve. It is not often the case that we will be able to solve for x explicitly, so the goal of Newton's Method is to instead *approximate* these solutions using a sequence.

Here is the picture to have in mind:



The value $x = a$ satisfying $f(a) = 0$ we are after is the point at which the graph of f intersects the x-axis. Take some number x_1 as a starting point, which is our first maybe not-so-good approximation to a. Consider the tangent line to the graph of f at $x = x_1$:



Denote by x_2 the point where this tangent line intersects the x-axis. The key observation is that, at least in the picture, this value x_2 is now a *better* approximation to a than x_1 was. To find what x_2 actually is, we recall that the tangent line to the graph of f at x_1 has the following equation:

$$y = f(x_1) + f'(x_1)(x - x_1)$$

with slope $f'(x_1)$. This intersects the x-axis when $y = 0$, which gives

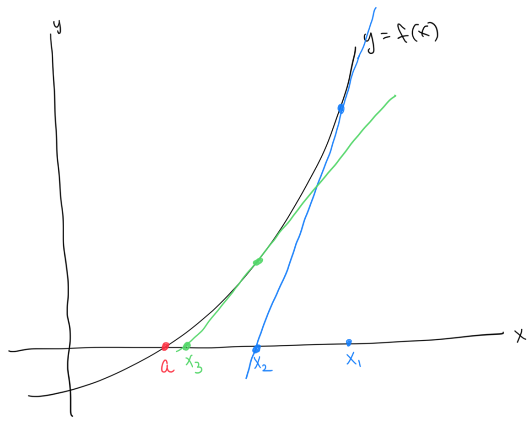
$$0 = f(x_1) + f'(x_1)(x - x_1), \text{ or } -f(x_1) = f'(x_1)(x - x_1).$$

The point x_2 is the value of x which satisfies this equation, so solving for x gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

(We will assume that $f'(x_1) \neq 0$ so that this expression makes sense; indeed, if $f'(x_1) = 0$ the tangent line is horizontal and hence would not actually intersect the x-axis. Newton's Method does not work if $f'(x_1) = 0$, in which case we should pick a different x_1 as our starting value.)

Now consider the tangent line to the graph of f at the point x_2:



The point where this tangent line intersects the x-axis is what we'll call x_3, with the point being that *this* is now an even better approximation to the number a we want than x_2 was. This tangent line has equation

$$y = f(x_2) + f'(x_2)(x - x_2),$$

and solving for the x-intercept just as before will give

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

And so on and so on, we continue this process, at each step taking the x-intercept of the tangent line corresponding to the previously obtained point to get a whole sequence of numbers $x_1, x_2, x_3, x_4, \ldots$. Concretely, all terms beyond the first are characterized by the equality

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which recursively defines each term in the sequence we are constructing in terms of the previous ones. The upshot is that—under some assumptions we'll take for granted hold in the examples we'll look at—the numbers $x_1, x_2, x_3, x_4, \ldots$ do indeed approximate a solution a of $f(x) = 0$, with the approximation becoming more accurate the further along we go.

Example. We use Newton's Method to approximate the positive solution of $x^2 - 7 = 0$, or in other words the positive root of the function $f(x) = x^2 - 7$. (A *root* of a function is just an input which results in the value 0.) Take $x_1 = 3$ as a starting point. Indeed, we know that number we actually want, $\sqrt{7}$ (i.e. the positive solution of $x^2 - 7 = 0$), should lie between 2 and 3 since

$$2^2 < 7 < 3^2,$$

so $x_1 = 3$ seems like a good first approximation.

Since $f'(x) = 2x$, the recursive equality defining the sequence in Newton's Method looks like:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 7}{2x_n}.$$

Thus, starting with $x_1 = 3$, we next get

$$x_2 = x_1 - \frac{x_1^2 - 7}{2x_1} = 3 - \frac{3^2 - 7}{2(3)} = 3 - \frac{2}{6} \approx 2.66667$$

after we round to the nearest fifth decimal location. We expect this value to be a better approximation to $\sqrt{7}$ than $x_1 = 3$. Now we use this value of x_2 to find the next term in Newton's Method:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.66667 - \frac{2.66667^2 - 7}{2(2.66667)} \approx 2.64583.$$

The next term is:

$$x_4 = 2.64583 - \frac{2.64583^2 - 7}{2(2.64583)} \approx 2.64575.$$

To summarize, the numbers we have obtained from Newton's Method so far are:

$$3,\ 2.66667,\ 2.64583,\ 2.64575,$$

each giving a better approximation to $\sqrt{7}$ than the last. In fact, the fourth term already gives the correct first five decimals in the decimal expression of $\sqrt{7}$, and any further terms we get from Newton's Method will only give an even better degree of accuracy.

Lecture 4: Infinite Series

Warm-Up. We approximate the solution of the equation $\cos x = x$ using Newton's Method. The point is that a number x satisfying this equality is the same as a number x satisfying $\cos x - x = 0$ instead, and this can be characterized as saying x is a root of the function $f(x) = \cos x - x$. So, we use Newton's Method to approximate this root.

The recursive sequence we get in Newton's Method is defined by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1}.$$

We can visualize the point we are looking for as the point at which the graphs of $y = \cos x$ and $y = x$ intersect, and by drawing these (say using a computer) we can eyeball that the intersection seems to occur near $x_1 = 1$, which we will take as our starting point in Newton's Method. Then we get:

$$x_2 = 1 - \frac{\cos(1) - 1}{-\sin(1) - 1} \approx 0.75.$$

Next we get:

$$x_3 = 0.75 - \frac{\cos(0.75) - 0.75}{-\sin(0.75) - 1} \approx 0.73911,$$

and then

$$x_4 = 0.73911 - \frac{\cos(0.73911) - 0.73911}{-\sin(0.73911) - 1} \approx 0.739085,$$

which turns out to give the correct value up to six decimal places.

One quibble you might have about our previous use of Newton's Method when approximating $\sqrt{7}$ is that Newton's Method was no necessary since, if we are going to use a calculator anyway, we may just as well have plugged in $\sqrt{7}$ to see the answer right away. One answer to this quibble is that the way your calculator actually computed $\sqrt{7}$ is likely through the use of Newton's Method itself, which is just hidden in the inner workings of whatever software was used. A better answer is that not all things one might want to approximate have easy answers like "$\sqrt{7}$" in terms of well-known expressions; indeed in this example, there is no way to easily write down what the solution of $\cos x = x$ is exactly, and approximating it using something like Newton's Method is the best we can do. This is the case for most equations of interest which pop-up in applications.

Why does Newton's Method Work? One final thing to clarify is the reason as to why Newton's Method works, in the sense that if the sequence it generates *does* converge, why it converges to a root of the given function: if x_n converges to L, why does L satisfy $f(L) = 0$? Take the recursive definition in Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We compute the limit of both sides as $n \to \infty$. On the left we get L again, since, as we saw in an earlier example, the terms in x_{n+1} are the same as those in x_n, only shifted over one spot, so they approach the same thing. The x_n term on the right also approaches L.

Now, f is continuous since otherwise it would not have a derivative, so since x_n converges to L, $f(x_n)$ converges to $f(L)$:

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(L).$$

If we also assume that f' is continuous (a standard assumption in Newton's Method), then $f'(x_n)$ will converge to $f'(L)$. Thus after taking limits of both sides in the recursive definition, we get:

$$L = L - \frac{f(L)}{f'(L)},$$

and solving for $f(L)$ indeed gives $f(L) = 0$. So, if the sequence Newton's Method generates does converge, it will definitely converge to a root of $f(x)$.

Series. And now, after having spent time developing the idea of a sequence, we can now talk about series, which are our main objects of interest this quarter. Recall that a *series* is an infinite sum, i.e. an expression where we attempt to add together infinitely many quantities. (To be clear, what makes this as "infinite" sum is the fact that we are adding infinitely many things, NOT that the resulting sum itself might be infinite.) We use the same $\sum$ notation for series we might have previously seen for Riemann sums when discussing integrals, only now we indicate the fact that our sums go on forever without end. To be clear, the notation

$$\sum_{n=0}^{\infty} a_n$$

denotes the infinite sum obtained by adding together all terms of the sequence a_n, starting at $n = 0$ and going beyond. So, in this case we get

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots ,$$

where the $\cdots$ indicate that our sum is without end. Series don't have to start at 0; for instance,

$$\sum_{n=2}^{\infty} a_n = a_2 + a_3 + a_4 + \cdots$$

begins the infinite sum at $n = 2$ instead. In general, the notion says we plug in the first value of n to get the first term, then increase n by 1 to compute the next term, then increase again and so on, adding on each new term we get at each step.

The key question we care about is whether a series *converges*, meaning that we actually get a finite value out of the given infinite sum, or *diverges*, meaning that we don't get a specific value. It is kind of amazing that even though we are adding together infinitely many quantities, we often get finite sums as a result.

Example 1. Consider the series

$$\sum_{n=1}^{\infty} n^2.$$

The first term when $n = 1$ is $1^2 = 1$, the second term when $n = 2$ is $2^2 = 4$, and so on. Writing out this series as an infinite sum gives

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + \cdots .$$

In this case, this infinite sum should intuitively not result in a finite value. One way to say this is that the terms we are adding on at each step are getting larger and larger, which in turn makes the resulting sum larger and larger. Another way to say this is to note that each term in our sum bigger than or equal to 1, so this given sum should be larger than the sum obtained by replacing each term by 1:

$$1 + 4 + 9 + 16 + \cdots \geq 1 + 1 + 1 + 1 + \cdots .$$

But adding together infinitely many 1's certainly results in ∞ as the value, so our sum, which is larger, should be infinite as well and hence should not converge, so it should diverge.

Partial Sums. But to make all of this precise, we have to be more careful about what it actually means for a series to converge. Consider the *partial sums* s_k of the series $\sum_{n=0}^{\infty} a_n$ in question, which are the sums obtained by adding one more term in our series at each step:

$$\begin{aligned}
s_0 &= a_0 \\
s_1 &= a_0 + a_1 \\
s_2 &= a_0 + a_1 + a_2 \\
s_3 &= a_0 + a_1 + a_2 + a_3 \\
&\vdots
\end{aligned}$$

In this notation, the k in s_k denotes the *last* term in the series we are adding on. We could also write this using $\sum$ notation as

$$s_k = a_0 + a_1 + \cdots + a_k = \sum_{n=0}^{k} a_n.$$

These numbers s_k form a new sequence, the *sequence of partial sums* of $\sum a_n$.

Now, if the infinite sum in question were to actually exist and equal S:

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + a_4 + \cdots = S,$$

the key point is that the partial sums a_k would provide better and better approximations to this value L: as we add on more terms in $a_0 + a_1 + a_2 + \cdots$ we should get closer and closer to number S which the entire infinite sum equals. So, we will take precisely this idea and use it to *define* what it means for the series to converge and have the value S:

> We say that the series $\sum_{n=0}^{\infty} a_n$ *converges* precisely when the sequence of partial sums $s_k = a_0 + \cdots + a_k$ converges, and if so we say that the limit S of this sequence of partial sums is the value of the infinite series: $\sum_{n=0}^{\infty} a_n = S$. The series $\sum_{n=0}^{\infty} a_n$ *diverges* if the sequence s_k of partial sums diverges.

So, the upshot is that asking whether or not a series converges to diverges *is exactly the same* as asking whether or not its sequence of partial sums converges or diverges. We will use this definition directly in a few examples, but soon enough we'll start to develop better *convergence tests* we can use to determine convergence or divergence of a series.

Warning. There are a lot of concepts and terms involved in saying that a series converges: first there is the sequence a_n, then the series $\sum a_n$ obtained by adding together the terms of a_n, and then another sequence $s_k = a_0 + \cdots + a_k$ formed by taking partial sums. These objects are all of course related, but they are *not* the same thing, so take care not to confuse them. In particular, asking whether or not a series $\sum a_n$ converges is NOT the same asking whether or not the sequence a_n converges, which is common point of confusion.

Back to Example 1. Back in Example 1 we mentioned the series $\sum_{n=1}^{\infty} 1$ obtained by adding together infinitely many 1's in relation to the series $\sum_{n=1}^{\infty} n^2$. We said that intuitively $\sum_{n=1}^{\infty} 1$ should diverge since adding together infinitely any 1's should not result in a definite, finite value. But now we can be more precise. We can compute the partial sums of $\sum_{n=1}^{\infty} 1$ as:

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + 1 = 2 \\ s_3 &= 1 + 1 + 1 = 3 \\ s_4 &= 1 + 1 + 1 + 1 = 4 \end{aligned}$$

and so on. In general, the k-th partial sum is $s_k = k$. Determining the convergence/divergence of the series $\sum_{n=1}^{\infty} 1$ is the same as determining the convergence/divergence of *this* sequence $s_k = k$ as $k \to \infty$, by definition of what it means for a series to converge. Since the sequence of partials sums $s_k = k$ diverges as $k \to \infty$, we thus know that the series $\sum_{n=1}^{\infty} 1$ diverges.

Now considering $\sum_{n=1}^{\infty} n^2$, its partial sums looks like:

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + 4 = 5 \\ s_3 &= 1 + 4 + 9 = 14 \\ s_4 &= 1 + 4 + 9 + 16 = 30 \end{aligned}$$

and so on. These partial sums are diverging since they only get larger and larger, and *this* is why the series $\sum_{n=1}^{\infty} n^2$ diverges. Again, the point is that the behavior of the partial sums determines the behavior of the series, *by definition*.

Example 2. Finally, we determine whether or not the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

converges. The point is that here we can also determine the partial sums explicitly. The k-th partial sum is the sum obtained by adding up the terms of the series only up to the $n = k$ term, so for instance the first partial sum is:

$$s_1 = 1 - \frac{1}{2},$$

the second partial sum is

$$s_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3},$$

the third partial sum is

$$s_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4},$$

and so on. In general, this pattern where intermediate terms cancel out carries through in all partial sums, so the k-th partial sum is

$$s_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{k+1}.$$

(This is what is known as a *telescoping series*, which is one where a piece of term cancels with a piece from a different term.) A series converges *by definition* when its sequence of partial sums converges, so since the sequence of partial sums $s_k = 1 - \frac{1}{k+1}$ in this case converges to 1 as k goes to infinity, our given series converges to 1. That is, the value of this infinite sum does exist and is equal to 1:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1.$$

Careful. The infinite sum we considered above looks like

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots .$$

If we regroup terms like so:

$$1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} - \frac{1}{4}\right) + \cdots ,$$

it might at first glance make sense to say that this is

$$1 + 0 + 0 + 0 + \cdots$$

since each term the parentheses in the regrouped expression is 0. This seems to suggest that the series should converge and indeed have the value 1. However, we have to be careful with this type of reasoning. For instance, instead consider the series

$$\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots .$$

Grouping terms like this:

$$(1-1) + (1-1) + (1-1) + \cdots$$

would suggest the value is $0 + 0 + 0 + 0 + \cdots = 0$, while grouping terms like this:

$$1 + (-1+1) + (-1+1) + (-1+1) + \cdots$$

suggests the value is $1 + 0 + 0 + 0 + \cdots = 1$. The series $\sum_{n=0}^{\infty}(-1)^n$ is actually divergent (i.e. not convergent) as we will see next time, so neither of these "values" is valid. The point is that trying to manipulate an infinite sum (by regrouping terms) as if it were a finite sum can lead to trouble. We really do have to rely on, for now, the behavior of the partial sums.

Lecture 5: More on Series

Warm-Up 1. We show that the series $\sum_{n=0}^{\infty}(-1)^n$ diverges. This makes sense intuitively since alternating between adding and subtracting 1's does not lead to a sum that approaches any one particular thing, but we need to look at the partial sums to be precise.

The partial sums of this series look like:

$$\begin{aligned}
s_0 &= (-1)^0 = 1 \\
s_1 &= 1 + (-1) = 0 \\
s_2 &= 1 + (-1) + 1 = 1 \\
s_3 &= 1 + (-1) + 1 + (-1) = 0
\end{aligned}$$

and so on. We get that the partial sums themselves alternate between 1 and 0: s_k is 1 when k is even and 0 when k is odd. Since this sequence of partial sums diverges (alternating 1's and 0's do not approach any one definite value), the series in question then diverges by definition.

Warm-Up 2. Now consider the series

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$$

The first few partial sums look like:

$$\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + 2 = 3 \\
s_3 &= 1 + 2 + 3 = 6 \\
s_4 &= 1 + 2 + 3 + 4 = 10.
\end{aligned}$$

In general, it turns out that adding together the first k positive integers gives the value $\frac{k(k+1)}{2}$:

$$s_k = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

(If you haven't seen this equality before, one way to derive it is to consider $2s_k = s_k + s_k$, only where we write the second s_k in the reverse order: $2s_k = (1 + 2 + \cdots + k) + (k + [k-1] + \cdots + 1)$, which can be written as $k+1$ added to itself k times by pairing the 1 in the first sum with the k in the second, the 2 in the first sum with the $k-1$ in the second, and so on. Thus $2s_k = k(k+1)$, and dividing by 2 gives the formula for s_k.) Thus since the sequence $s_k = \frac{k(k+1)}{2}$ of partial sums diverges as $k \to \infty$, the series $\sum_{n=1}^{\infty} n$ diverges as well.

First divergence test. The two Warm-Up examples already illustrate our first divergence test: in a series $\sum_{n=1}^{\infty} a_n$, if the *sequence* a_n does not converge to 0, the *series* $\sum_{n=1}^{\infty} a_n$ must diverge. Now, it is absolute crucial here to recognize what this is saying, which highlights the difference between sequences and series. Saying that the *series* $\sum_{n=1}^{\infty} a_n$ converges is NOT the same as saying that the sequence a_n converges! As emphasized last time, these are related concepts for sure, but they do not mean the same thing. The *sequence* a_n describes the individual terms being added together to produce the *series* $\sum a_n$.

What this first divergence test says is that if the terms of the sequence a_n do not themselves approach 0, the infinite sum

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

cannot possibly exist. In the previous example, the sequence n^2 does not converge to 0 (in fact it diverges), so the series $\sum_{n=1}^{\infty} n^2$ cannot converge. The intuitive idea is that in order for an infinite sum

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

to have any hope of resulting in a finite value, it had better be the case that the terms being added on at each step are getting smaller and smaller; if this is not the case, the infinite sum cannot actually exist as a finite value. This test goes by various names: our book calls it the "n-th term test", and other books call it the "test for divergence".

Warning. So, possibly the first thing to do when considering whether a series $\sum a_n$ converges or diverges is to see what is happening with the sequence a_n: if $\lim_{n\to\infty} a_n \neq 0$, you're finished—the series $\sum a_n$ will diverge. But as a warning (also a common cause of confusion): just because $\lim_{n\to\infty} a_n$ does equal 0 does NOT mean that the corresponding series $\sum a_n$ converges! Knowing that a_n converges to 0 only tells us that the series $\sum a_n$ has some hope of converging, but does not by itself tell us that the series does indeed converge; this is why we need to consider further convergence tests, as we'll develop in the coming days. Again, the convergence of the *sequence* a_n is related to the convergence of the series $\sum a_n$, but it is not literally the same idea; sequences and series are related but different concepts!

Geometric series. One of the most important types of series is what is known as a *geometric series.* This is a series of the form

$$\sum_{n=0}^{\infty} r^n,$$

where we are adding together higher and higher powers of a number r. So, written as an infinite sum, a geometric series looks like

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \cdots.$$

The first term of 1 comes from $r^0 = 1$. Geometric series are important because it is often the case that other series can be related to or compared with such a series, which is good because we can tell exactly when a geometric series converges *and* what it converges to when it does!

Here is the basic fact you should know by heart: when r is a number outside of the interval $(-1, 1)$ (so when $r \leq -1$ or $1 \leq r$), the geometric series $\sum_{n=0}^{\infty} r^n$ diverges, while when r is a number in the interval $(-1, 1)$ (so when $-1 < r < 1$), the geometric series $\sum_{n=0}^{\infty} r^n$ converges to the value $\frac{1}{1-r}$. We write this as

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \text{ when } -1 < r < 1.$$

Amazingly this is saying that if we add up the numbers $1, r, r^2, r^3, r^4, \ldots$ we get $\frac{1}{1-r}$ as the result, at least when $-1 < r < 1$, even though there are infinitely many numbers we are adding together. So, this is a concrete example of an infinite sum which actually has a finite value.

So, how do we know that $\sum_{n=0}^{\infty} r^n$ converges when $-1 < r < 1$ and that its value in this case is $\frac{1}{1-r}$? The answer, of course, comes from the partial sums. The first few partial sums are:

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 + r \\ s_2 &= 1 + r + r^2 \\ s_3 &= 1 + r + r^2 + r^3 \end{aligned}$$

so $s_k = 1 + r + r^2 + \cdots + r^k$ in general. It turns out there is a simpler way of expressing such a sum. For instance, when $k = 1$ we have

$$1 + r = \frac{1 - r^2}{1 - r},$$

which we can see by factoring $1 - r^2$ into $(1 - r)(1 + r)$. When $k = 2$ we have

$$1 + r + r^2 = \frac{1 - r^3}{1 - r},$$

which can see by noting that $(1 + r + r^2)(1 - r) = 1 - r^3$. In general, the k-th partial sum can be written as

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}.$$

Indeed, you can check that $(1 + r + r^2 + \cdots + r^k)(1 - r)$ does multiply out to $1 - r^{k+1}$, and dividing through by $1 - r$ gives the formula above.

Now we're in business! Recall that r was a number between -1 and 1. For such numbers, we have

$$\lim_{k \to \infty} r^{k+1} = 0.$$

Thus the limit of the k-th partial sums as k goes to ∞ is:

$$\lim_{k \to \infty} \frac{1 - r^{k+1}}{1 - r} = \frac{1 - 0}{1 - r} = \frac{1}{1 - r},$$

so in this case the series $\sum_{n=0}^{\infty} r^n$ converges and has the value $\frac{1}{1-r}$. For a number r which is not between -1 and 1, $\lim_{k \to \infty} r^k$ is not zero, and then the partial sum $s_k = \frac{1-r^{k+1}}{1-r}$ diverges, so the geometric series $\sum_{n=0}^{\infty} r^n$ diverges when r is outside $(-1, 1)$.

Example. We show that the series

$$\sum_{n=0}^{\infty} \frac{2 \cdot 3^{n+1}}{5^n}$$

converges and determine its value. The key is that we can express this series in terms of a geometric series. By rewriting the terms we are adding together, we get:

$$\sum_{n=0}^{\infty} \frac{2 \cdot 3^{n+1}}{5^n} = \sum_{n=0}^{\infty} \frac{2 \cdot 3^n \cdot 3}{5^n} = \sum_{n=0}^{\infty} 6 \left(\frac{3}{5}\right)^n.$$

The point of doing this is to obtain an expression which involves taking a fixed number to the n-th power, since this is the type of expression which shows up in a geometric series. After "factoring out" the 6, we're left with a geometric series with $\frac{3}{5} < 1$, so

$$\sum_{n=0}^{\infty} \frac{2 \cdot 3^{n+1}}{5^n} = \sum_{n=0}^{\infty} 6 \left(\frac{3}{5}\right)^n = 6 \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

converges. The value of the series above before we multiply by 6 is found using

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

with $r = \frac{3}{5}$; we get $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{1}{1-\frac{3}{5}}$. Hence after multiplying by 6 and simplifying we see that the series in question converges to 15:

$$\sum_{n=0}^{\infty} \frac{2 \cdot 3^{n+1}}{5^n} = 6 \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = 6 \left(\frac{1}{1-\frac{3}{5}}\right) = 6 \frac{1}{\frac{2}{5}} = \frac{30}{2} = 15.$$

To clarify one point: why is that we can "factor out" 6 as we did above? The point is that this is simply a version of the *distributive* property of multiplication. If we write out the terms of the sum

$$\sum_{n=0}^{\infty} 6 \left(\frac{3}{5}\right)^n$$

we get:

$$6 + 6\left(\frac{3}{5}\right) + 6\left(\frac{3}{5}\right)^2 + 6\left(\frac{3}{5}\right)^2 + \cdots.$$

Here we can factor the 6 out to get:

$$6\left(1 + \frac{3}{5} + \frac{3^2}{5^2} + \frac{3^3}{5^3} + \cdots\right),$$

which can be written back in terms of summation notation as:

$$6 \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n.$$

The point, again, is that this is just an infinite version of the distributive property.

Alternatively, imagine we had the same series only with a different starting point:

$$\sum_{n=1}^{\infty} 6\left(\frac{3}{5}\right)^n.$$

How do we determine the value of this series? If we write out these terms we get:

$$6\left(\frac{3}{5}\right)+6\left(\frac{3}{5}\right)^2+6\left(\frac{3}{5}\right)^3+\cdots.$$

The key observation is that this sum is the same as the one we had above:

$$6+6\left(\frac{3}{5}\right)+6\left(\frac{3}{5}\right)^2+6\left(\frac{3}{5}\right)^2+\cdots$$

only that the initial 6 (the zeroth) term is missing. So, this new sum should be equal to the old one minus 1:

$$\sum_{n=1}^{\infty} 6\left(\frac{3}{5}\right)^n=\sum_{n=0}^{\infty} 6\left(\frac{3}{5}\right)^n-1.$$

We found the value of the old one above to be 15, so the value of this new series is 24:

$$\sum_{n=1}^{\infty} 6\left(\frac{3}{5}\right)^n=\sum_{n=0}^{\infty} 6\left(\frac{3}{5}\right)^n-1=25-1=24.$$

In general, since

$$\sum_{n=0}^{\infty} a_n=a_0+a_1+a_2+\cdots=a_0+\sum_{n=1}^{\infty} a_n,$$

we get

$$\sum_{n=1}^{\infty} a_n=\sum_{n=0}^{\infty} a_n-a_0.$$

A similar thing works for other sums that start at a value larger than 1; for instance:

$$\sum_{n=2}^{\infty}=\sum_{n=0}^{\infty} a_n-a_0-a_1.$$

Lecture 6: Integral Test

Warm-Up 1. Consider the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{4^{n+2}}{3^{n-1}}.$$

We claim that this diverges, which we can show by expressing it in terms of a geometric series. We can rewrite the n-term in the sequence being summed up as:

$$(-1)^n\frac{4^{n+2}}{3^{n-1}}=\frac{(-1)^n4^n4^2}{3^n3^{-1}}=48\left(-\frac{4}{3}\right)^n.$$

To be clear, the 48 came from $\frac{4^2}{3^{-1}} = 16 \cdot 3$. Thus our given series is

$$\sum_{n=1}^{\infty} (-1)^n \frac{4^{n+2}}{3^{n-1}} = \sum_{n=1}^{\infty} 48 \left(-\frac{4}{3}\right)^n = 48 \sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)^n.$$

But now the resulting geometric series $\sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)^n$ diverges since $-\frac{4}{3}$ falls outside the interval $(-1, 1)$, so the original series diverges as well.

Warm-Up 2. We now look at the series

$$\sum_{n=2}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right).$$

This is not itself a geometric series, but it can be expressed as the sum of two geometric series, namely

$$\sum_{n=2}^{\infty} \frac{1}{2^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{3^n} = \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n.$$

Each of these converges since the respective values we are taking powers of, $\frac{1}{2}$ and $\frac{1}{3}$, are each between -1 and 1. It is a fact that sum of two convergent series is itself convergent, so our given series is convergent.

To find the value of our series we find the value of each component series first. We have:

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^0 - \left(\frac{1}{2}\right)^1 = \frac{1}{1-\frac{1}{2}} - 1 - \frac{1}{2} = 2 - 1 - \frac{1}{2} = \frac{1}{2},$$

where we use the fact that the series starting at $n = 2$ is *almost* the series starting at $n = 0$ except that it excludes the term corresponding to $n = 0$ and the term corresponding to $n = 1$:

$$a_2 + a_3 + a_4 + \cdots = (a_0 + a_1 + a_2 + a_3 + a_4 + \cdots) - a_0 - a_1.$$

Similarly, we have:

$$\sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^0 - \left(\frac{1}{3}\right)^1 = \frac{1}{1-\frac{1}{3}} - 1 - \frac{1}{3} = \frac{3}{2} - 1 - \frac{1}{3} = \frac{1}{6}.$$

Thus, overall we find that

$$\sum_{n=2}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n}\right) = \sum_{n=2}^{\infty} \frac{1}{2^n} + \sum_{n=2}^{\infty} \frac{1}{3^n} = \frac{1}{2} + \frac{1}{6}.$$

Just one final observation. Note it makes intuitive sense to say that

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \frac{1}{2}$$

as we derived above by interpreting what this means in terms of distance: start at the point 0 on a number line and walk a distance of $\frac{1}{4}$ (half the way to $\frac{1}{2}$), then another distance of $\frac{1}{8}$ (half the remaining distance to $\frac{1}{2}$), then another distance of $\frac{1}{16}$ (half the remaining distance to $\frac{1}{2}$), and so on and so on; if we keep moving half the remaining distance to 2 at each step forever and ever, we will overall (in an infinite amount of time) reach $\frac{1}{2}$ itself, which is what the equality above says.

Convergence tests. The main question we will be interested in for now is determining whether or not a series converges. So far we can only easily answer this for geometric series, or possibly examples where a formula for the partial sums can be determined explicitly, such as for a telescoping series. But most series are not of these types, so we need more convergence tests. This is what we'll focus on over the next few lectures; in particular we'll look at the Integral Test, the Comparison Test, the Limit Comparison Test, the Alternating Series Test, and the Ratio Test. It is crucial to learn to recognize the types of series each of these tests is best suited for. Our eventual goal is to understand how we can represent functions as series, but the point is that first we need to better understand when series actually converge.

None of the tests we'll consider will give us a way to determine the actual value of a convergent series, only a way to determine that a series does indeed converge. Finding values is something we'll be able to do more easily after we discuss the idea of expressing a function as a series.

Integral test. The integral test applies to series of the form

$$\sum_{n=1}^{\infty} f(n)$$

where f is a continuous, positive, *decreasing* function. The integral test says that in this setting, the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\,dx$ both behave in the same way, meaning they both converge or they both diverge. This is useful since it is usually simpler to determine whether or not an improper integral converges since we can often try to compute its actual value. So, we can turn problems about series convergence into ones about integral convergence instead. Note that the integral test does NOT say that

$$\sum_{n=1}^{\infty} f(n) \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

have the same value, only that both values are either finite or infinite at the same time. We'll give some intuition behind the integral test after the first example below.

Example 1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

This is known as the *harmonic* series. The terms $\frac{1}{n}$ in our series come from plugging n into the function

$$f(x) = \frac{1}{x}.$$

Since this function is positive and decreasing, the integral test applies. (Note that this function is decreasing since increasing x makes the value $f(x)$ smaller. Another way to see this is that the derivative $f'(x) = -\frac{1}{x^2}$ is always negative.) So, we instead consider the integral

$$\int_1^{\infty} \frac{1}{x}\,dx,$$

which we can work with directly. We get:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x\Big|_1^b = \lim_{b\to\infty}(\ln b - 1) = \infty.$$

Thus $\int_1^\infty f(x)\,dx$ diverges, so the integral test says that the series $\sum_{n=1}^\infty \frac{1}{n}$ diverges as well.

Note here that even though the terms of this series $\frac{1}{n}$ approach zero, the *series* obtained by adding these terms together does NOT converge. We've mentioned previously that if $\lim_{n\to\infty} a_n$ is not zero, then $\sum a_n$ for sure diverges, and this now an examples where $\lim_{n\to\infty} a_n = 0$ and yet $\sum a_n$ diverges. In general, knowing that $\lim_{n\to\infty} a_n = 0$ says *nothing* about whether or not $\sum a_n$ convergence, so a different convergence test is needed. The point of this example is that even though the terms being added on at each step in

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

are getting smaller and smaller, the sum of *all* of them is actually infinite. (Surprising, no?)

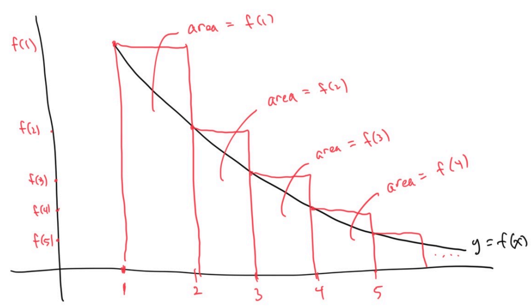
Why does the integral test work? To get a glimpse as to why the integral test works, we note that the series

$$\sum_{n=1}^\infty f(n)$$

in the setup of the integral test can be interpreted as a sum of areas. Indeed, think of each $f(n)$ term as

$$f(n)\cdot 1,$$

which is the area of a rectangle of height $f(n)$ and base length 1. Draw these rectangles like so, where the n-th one has base given by the interval $[n, n+1]$:



The sum of the areas of these (infinitely many) rectangles is:

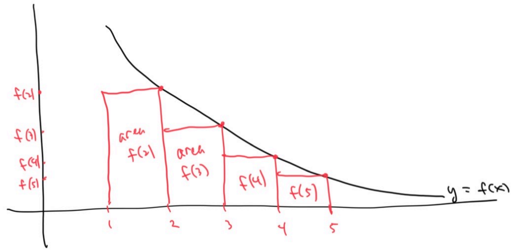
$$f(1) + f(2) + f(3) + f(4) + \cdots = \sum_{n=1}^\infty f(n).$$

The integral test is saying that this sum of areas is finite if and only if the area under the curve $y = f(x)$ from $x = 1$ to infinity is finite.

Based on the picture above, we can see that the sum of the areas of the rectangles is actually larger than the area under the curve, so:

$$\sum_{n=1}^\infty f(n) \geq \int_1^\infty f(x)\,dx \geq 0.$$

Thus if the integral is infinite, so is the series and hence it diverges. Now, from this picture below:



we see that $0 \leq \sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)\,dx$, so if the integral is finite so is the series. (Technically, this picture will imply that $\sum_{n=2}^{\infty} f(n)$ converges, but this in turn implies that the series starting at $n = 1$ also converges.) Again the idea is that even though these values aren't the same, they are pretty close to one another so that either both are finite or both are infinite simultaneously, which is the statement of the integral test. Again, this is not a proof, but is meant to suggest that there should be some relation between the value of $\sum_{n=1}^{\infty} f(n)$ and that of $\int_1^{\infty} f(x)\,dx$.

p-**series test.** The same type of integral computation used in the example above applies to series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where p is some constant. (The previous example was the case $p = 1$.) The function $f(x) = \frac{1}{x^p}$ is positive and decreasing on the interval $[1, \infty)$, so the integral test applies to say that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } \int_1^{\infty} \frac{1}{x^p}\,dx \text{ converges.}$$

For $p > 1$, the corresponding integral is:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \frac{1}{(1-p)x^{p-1}}\bigg|_1^b = \lim_{b\to\infty} \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right) = \frac{1}{p-1}.$$

It is important that $p > 1$ since otherwise b^{p-1} would not remain in the denominator of the fraction in the final limit; if instead $p < 1$, we end up with the limit of $\frac{1}{1-p}(b^{1-p} - 1)$ as $b \to \infty$, which is ∞ since $1 - p > 0$. Hence in this case the integral diverges.

Thus we get that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1, \text{ and diverges if } p \leq 1.$$

Such series are known as *p-series*, and so we'll call this is the *p*-series test.

Fun fact. Here is an interesting fact, which goes way beyond the scope of this course: the actual value of the *p*-series above when $p = 2$ is $\frac{\pi^2}{6}$:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}.$$

That is, if you could literally add up all the infinitely many terms on the left, you would get $\frac{\pi^2}{6}$ as the result. This is surprising since it is in no way clear what π has to do with the sum of such fractions, and the π seems to come out of nowhere. You might see a justification of this value if you ever learn about *Fourier series*, but this is not something we'll come back to in this course. So, no, you do NOT have to know that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ for the purposes of an exam; this was only meant to illustrate an interesting observation.

Example 2. Finally we consider the series

$$\sum_{n=2}^\infty n^2 e^{-n^3}.$$

First, the fact that we are starting at $n = 2$ instead of $n = 1$ is not important: in this case we simply use the integral test with the integral $\int_2^\infty x^2 e^{-x^3}\,dx$ instead. The fact that $x^2e^{-x^3}$ is possible to integrate (using the substitution $u = -x^3$) without too much trouble is what suggests the integral test may be useful. To be sure the integral test applies, we consider the function

$$f(x) = x^2 e^{-x^3}.$$

This is positive for $x \geq 2$, and

$$f'(x) = 2xe^{-x^3} + x^2e^{-x^3}(-3x^2) = (2x - 3x^4)e^{-x^3}$$

is negative for $x \geq 2$ since $2x - 3x^4$ is negative but e^{-x^3} is positive. The function $f(x)$ is also continuous, so the integral test is applicable.

We have

$$\begin{aligned}\int_2^\infty x^2e^{-x^3}\,dx &= \lim_{b\to\infty}\int_2^b x^2e^{-x^3}\,dx \\ &= \lim_{b\to\infty} -\frac{1}{3}e^{-x^3}\Big|_2^b \\ &= \lim_{b\to\infty} -\frac{1}{3}(e^{-b^3} - e^{-8}) \\ &= \frac{1}{3}e^{-8}.\end{aligned}$$

(To be clear, to evaluate the integral $\int_2^b x^2e^{-x^3}\,dx$ we used the substitution $u = -x^3$.) Thus $\int_2^\infty x^2e^{-x^3}\,dx$ converges, so the integral test says that $\sum_{n=2}^\infty n^2e^{-n^3}$ converges as well.

Lecture 7: Comparison Tests

Warm-Up 1. We determine whether the series

$$\sum_{n=3}^\infty \frac{n^2}{n^3+1}$$

converges or diverges. The fact that $\frac{x^2}{x^3+1}$ is possible to integral using a substitution suggests that the integral test may be useful. (The integral test is not the only thing which works here; we'll see later that we can also use the *limit comparison test.*) First we verify that the integral test is actually applicable. The function

$$f(x) = \frac{x^2}{x^3+1}$$

is positive for $x \geq 3$, and since

$$f'(x) = \frac{(x^3+1)2x - x^2(3x^2)}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} = \frac{2x(1-x^3)}{(x^3+1)^2}$$

is negative for $x \geq 3$ (since the numerator is negative and the denominator positive), the integral test is indeed applicable.

We have:

$$\begin{aligned}
\int_3^\infty \frac{x^2}{x^3+1}\,dx &= \lim_{b\to\infty} \int_3^b \frac{x^2}{x^3+1}\,dx \\
&= \lim_{b\to\infty} \frac{1}{3} \ln|x^3+1| \Big|_3^b \\
&= \lim_{b\to\infty} \frac{1}{3}[\ln(b^3+1) - \ln 10] \\
&= \infty,
\end{aligned}$$

where the integral $\int_3^b \frac{x^2}{x^3+1}\,dx$ was computed using the substitution $u = x^3+1$. Thus $\int_3^\infty \frac{x^2}{x^3+1}\,dx$ diverges, so the integral says that $\sum_{n=3}^\infty \frac{n^2}{n^3+1}$ diverges as well.

Warm-Up 2. We determine whether the series

$$\sum_{n=2}^\infty \frac{1}{n \ln n}$$

converges or diverges. The function

$$f(x) = \frac{1}{x \ln x}$$

is positive for $x \geq 2$, and

$$f'(x) = -\frac{\ln x + 1}{(x \ln x)^2}$$

is negative for $x \geq 2$, so f is decreasing. Hence the integral test is applicable.

We have:

$$\begin{aligned}
\int_2^\infty \frac{1}{x \ln x}\,dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x}\,dx \\
&= \lim_{b\to\infty} \ln|\ln x| \Big|_2^b \\
&= \lim_{b\to\infty} (\ln\ln b - \ln\ln 2) \\
&= \infty,
\end{aligned}$$

so $\int_2^\infty \frac{1}{x\ln x}\,dx$ diverges. Hence $\sum_{n=2}^\infty \frac{1}{n \ln n}$ diverges by the integral test. (The integral of $\frac{1}{x \ln x}$ was computed using the substitution $u = \ln x$.)

If instead we considered the series

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^2},$$

the same technique would apply. The function $g(x) = \frac{1}{x(\ln x)^2}$ is still positive and decreasing (which can be checked by seeing that the derivative is negative), so the integral test applies. In this case we have:

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{\ln x}\Big|_2^b = \lim_{b\to\infty} -\left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Hence $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$ converges, so $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges as well by the integral test.

Direct comparison test. The direct comparison test (and the limit comparison test we'll look at next) allows us to determine whether or not a series converges by *comparing* it to a simpler series whose convergence/divergence is simpler to determine. This test applies to series consisting of all positive terms, so to something like

$$\sum_{n=0}^\infty a_n \text{ where each } a_n \text{ is positive.}$$

The key facts to remember are:

> if the larger series converges, so does the smaller one; and, if the smaller series diverges, so does the larger one.

The point is that such a series of positive terms definitely cannot result in a negative value, so the value is either some positive number or is infinite. So, the only thing we need to determine is whether or not the series has an infinite value (so diverges) or a finite value (so converges).

The first thing we need to do with such series is come up with a *guess* as to whether it should converge or diverge, so that we know which type of comparison we'll need to apply. This is also important because the way in which we come up with this guess will often suggest with which series we should compare our given one. Let's work out some examples to see how this works.

Example 1. Consider the series

$$\sum_{n=1}^\infty \frac{10n^2-3n-1}{n^4+n^2+1}.$$

First we need a guess. The things to focus on are the "dominant" terms in the sequence

$$\frac{10n^2-3n-1}{n^4+n^2+1}.$$

In this case, the dominant term (i.e. the term that overpowers everything else) in the numerator is $10n^2$, while the dominant term in the denominator is n^4. This suggests that, roughly, our given series should behave in a "similar" way to the series

$$\sum_{n=1}^\infty \frac{10n^2}{n^4} = \sum_{n=1}^\infty \frac{10}{n^2}.$$

This latter series converges by the p-series test, so we make an educated guess that our given series does as well.

Now to make this guess actually precise, we can make a comparison. Note that

$$\frac{10n^2-3n-1}{n^4+n^2+1} \le \frac{10n^2}{n^4} = \frac{10}{n^2},$$

where the inequality came from making the numerator bigger and the denominator smaller. This equality implies that

$$0 \leq \sum_{n=1}^{\infty} \frac{10n^2 - 3n - 1}{n^4 + n^2 + 1} \leq \sum_{n=1}^{\infty} \frac{10}{n^2}.$$

Since the sum on the right is finite (since $\sum_{n=1}^{\infty} \frac{10}{n^2}$ converges), the first infinite sum should be finite as well. (This is what it means to say that "if the larger series converges, so does the smaller one".) Thus

$$\sum_{n=1}^{\infty} \frac{10n^2 - 3n - 1}{n^4 + n^2 + 1}$$

converges by the direct comparison test. Note that the series $\sum \frac{10}{n^2}$ we compared our given one with wasn't just pulled out of thin air, but rather came from the series we used in our guess.

Example 2. Consider now the series

$$\sum_{n=2}^{\infty} \frac{10n^4 + n^2 + n + 1}{n^5 - n^4 - 3}.$$

The dominant term in the numerator is $10n^4$ and in the denominator it is n^5. Thus our series should behave in a manner roughly similar to

$$\sum_{n=2}^{\infty} \frac{10n^4}{n^5} = \sum_{n=2}^{\infty} \frac{10}{n},$$

which diverges since it is just 10 times the divergent series $\sum_{n=2}^{\infty} \frac{10}{n}$. So, we guess that our given series diverges.

To show that it diverges using the comparison test, we have to find a "smaller' series which diverges. (In the first example we wanted to show that the given series converged, which meant we had to compare it with a *larger* series which converged.) In this case we have:

$$\frac{10n^4}{n^5} \leq \frac{10n^4 + n^2 + n + 1}{n^5 - n^4 - 3}$$

since the fraction on the left has a smaller numerator and larger denominator than the one on the right. This implies that

$$0 \leq \sum_{n=2} \frac{10n^4}{n^5} \leq \sum_{n=2}^{\infty} \frac{10n^4 + n^2 + n + 1}{n^5 - n^4 - 3},$$

so since the smaller series $\sum_{n=2}^{\infty} \frac{10}{n}$ diverges, so does the larger one. Hence

$$\sum_{n=2}^{\infty} \frac{10n^4 + n^2 + n + 1}{n^5 - n^4 - 3}$$

diverges by the direct comparison test. Again, note that series we used to compare our given one to came from our educated guess.

Example where direct comparison doesn't quite work. Consider the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + n + 1}.$$

Focusing on dominant terms suggests that this series should diverge since

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. If we wanted to use the direct comparison test to show that our original series diverged, we would have to find a smaller series which diverged. The simplest inequality we can use given the terms of our series is

$$\frac{n^2}{n^3+n+1} \leq \frac{n^2}{n^3} = \frac{1}{n},$$

which comes from making the denominator smaller. However, this does us no good: in this case, the divergent series $\sum \frac{1}{n}$ is the *larger* one, and knowing that the larger series diverges tells us nothing about the smaller one. So, doing an ordinary comparison between our original series and $\sum \frac{1}{n}$ leads us nowhere.

However, the guess that our given series should diverge since it should be similarly to $\sum \frac{1}{n}$ was a good one, we just need another way to make this precise. Here is where the *limit comparison test* comes in; this is also a way to compare a given series with another, but where we don't have to worry about which series is "larger" and which is "smaller". To match up with the notation we'll use in a second when stating the limit comparison test in general, denote the terms of our given series and the one we're comparing it to by:

$$a_n = \frac{n^2}{n^3+n+1} \quad \text{and} \quad b_n = \frac{1}{n}.$$

We compute the limit $\lim_{n\to\infty} \frac{a_n}{b_n}$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^3+n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}+\frac{1}{n^3}} = 1,$$

where the third equality came from multiplying numerator and denominator by $\frac{1}{n^3}$. Since this limit exists and is positive, the limit comparison test says that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2}{n^3+n+1} \text{ indeed diverges since } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

Limit comparison test. Here is the statement. First, as with the direct comparison and integral tests, the limit comparison test only applies to series consisting of positive terms. For such series $\sum a_n$ and $\sum b_n$, look at the limit

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

The *limit comparison test* says that if this limit exists and is positive, then both series $\sum a_n$ and $\sum b_n$ behave in the same way, meaning they both converge or they both diverge. Compared to the direct comparison test, we still need to come up with a series to which we can compare our given one, but the benefit is that now we don't have to work with inequalities.

Intuition behind limit comparison. This is not something you have to know, but it's nice to get a sense for why the limit comparison test works. The idea is that we can think of

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} \text{ existing and being positive}$$

as saying that $\frac{a_n}{b_n}$ should get closer and closer to L as n gets larger, so that (after multiplying through by b_n)

$$a_n \text{ itself gets closer and closer to behaving like } Lb_n.$$

This suggests that

$$\sum a_n \text{ should behave similar to how } \sum Lb_n = L\sum b_n \text{ behaves,}$$

so either both $\sum a_n$ and $\sum b_n$ give finite values (i.e. converge), or both give infinite values (i.e. diverge). Again, this is not an actual proof, just some intuition.

Lecture 8: Ratio Test & Absolute Convergence

Warm-Up 1. Consider the series

$$\sum_{n=5}^{\infty} \frac{e^{-n}}{n^2+3}.$$

The e^{-n} term is getting smaller and smaller as n increases, so, if we think of this term as $1 \cdot e^{-n}$, in a sense the "dominant" term in the numerator is 1. In the denominator the dominant term is n^2, so the series should behave similarly to

$$\sum_{n=5}^{\infty} \frac{1}{n^2},$$

which converges by the p-series test. Thus we guess that our original series converges.

Since we want to show the original series converges, if we want to apply the comparison test we need to come up with a larger series which converges. We have:

$$\frac{e^{-n}}{n^2+3} \leq \frac{1}{n^2},$$

which we can see is true by noting that the numerator on the right is larger than the one on the right and the denominator is smaller. Thus since the larger series $\sum_{n=5}^{\infty} \frac{1}{n^2}$ converges, so does the

$$\sum_{n=5}^{\infty} \frac{e^{-n}}{n^2+3}$$

converges by the comparison test.

Warm-Up 2. Consider the series

$$\sum_{n=5}^{\infty} \frac{n(1+e^{-n})}{n^3+3}.$$

All of the terms in this series are positive, and focusing on dominant terms suggests that this series should behave similarly to

$$\sum_{n=5}^{\infty} \frac{n}{n^3} = \sum_{n=5}^{\infty} \frac{1}{n^2},$$

which converges. Thus we guess that our given series should converge too.

To show this, we use the limit comparison test with the series $\sum \frac{1}{n^2}$ we used in our guess. We look at the limit:

$$\lim_{n\to\infty} \frac{\frac{n(1+e^{-n})}{n^3+3}}{\frac{1}{n^2}},$$

which is $\lim_{n\to\infty} \frac{a_n}{b_n}$ in the case where a_n denote the terms of our given series and b_n the terms of the one we are comparing it to. After simplifying, we get:

$$\lim_{n\to\infty} \frac{\frac{n(1+e^{-n})}{n^3+3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3(1+e^{-n})}{n^3+3} = \lim_{n\to\infty} \frac{1+e^{-n}}{1+\frac{3}{n^3}} = 1.$$

Thus $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists (meaning it is finite) and is positive, so since $\sum \frac{1}{n^2}$ converges, the series

$$\sum_{n=5}^{\infty} \frac{n(1+e^{-n})}{n^3+3}$$

converges as well by the limit comparison test.

Careful. When $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, the limit comparison test may or may not give us enough information to determine convergence. For instance, consider the series

$$\sum_{n=5}^{\infty} \frac{e^{-n}}{n^2+3}$$

we saw in the first Warm-Up. If we try to use a limit comparison with the series $\sum_{n=5}^{\infty} \frac{1}{n^2}$, we would get:

$$\lim_{n\to\infty} \frac{\frac{e^{-n}}{n^2+3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2 e^{-n}}{n^2+3} = \lim_{n\to\infty} \frac{e^{-n}}{1+\frac{3}{n^2}} = 0.$$

Since we got a limit of zero for $\lim_{n\to\infty} \frac{a_n}{b_n}$, the limit comparison test we described does not apply. However, there is actually a version of the limit comparison test which does apply here, and would say that since $\sum \frac{1}{n^2}$ converges, so does our series. Similarly, there is a version of the limit comparison test which applies when $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. We did not look at these in class, but you can find them in the book. On an exam you should not expect that one of these alternate versions will be necessary and some other test will be applicable instead; for instance, in the example above, the direct comparison test (as we used in the Warm-Up) works just fine.

This suggests that there is not always one single test tor try: sometimes one test works when another doesn't, and sometimes multiple tests work. As I said last time, getting used to recognizing which type of test to use in which scenarios is something which comes with much practice.

Ratio test. Our next convergence test is called *ratio test*, which is possibly the simplest one to apply when it works. Say we have a series $\sum_{n=1}^{\infty} a_n$. We look at the limit:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

obtained from the absolute value of the fraction of the $(n+1)$-st term in our series and the n-term. The ratio test says that:

- if $L < 1$, then $\sum a_n$ converges (actually, it converges *absolutely*, which is better; we'll define this term in a bit)

- if $L > 1$ or $L = \infty$, then $\sum a_n$ diverges, and
- if $L = 1$, the ratio test gives no information.

Thus, computing the required limit will tells us whether or series converges or diverges, as long as the limit is not equal to 1.

Example 1. Consider the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{5^n}.$$

In the notation of the ratio test, $a_n = (-1)^n \frac{n}{5^n}$, so we compute the limit:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(-1)^{n+1}\frac{n+1}{5^{n+1}}|}{|(-1)^n \frac{n}{5^n}|} = \lim_{n\to\infty} \frac{(n+1)5^n}{n5^{n+1}} = \lim_{n\to\infty} \frac{(n+1)}{5n} = \frac{1}{5}.$$

Note that the $(-1)^n$ terms disappeared after taking absolute values. Thus since this limit is less than 1, the ratio test tells us that this series converges. (Actually, as stated above, it tells us that it converges absolutely.)

Example 2. Next we consider

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}.$$

In the ratio test we need the following limit:

$$\lim_{n\to\infty} \frac{|\frac{(-3)^{n+1}}{(2(n+1)+1)!}|}{|\frac{(-3)^n}{(2n+1)!}|} = \lim_{n\to\infty} \frac{3^{n+1}}{(2n+3)!}\frac{(2n+1)!}{3^n} = \lim_{n\to\infty} \frac{3}{(2n+3)(2n+2)} = 0.$$

To be clear, here we used the fact that

$$(2n+3)! = (2n+3)(2n+2)(2n+1)(2n)\cdots 2\cdot 1 \text{ and } (2n+1)! = (2n+1)(2n)\cdots 2\cdot 1$$

in order to simplify:

$$\frac{(2n+1)!}{(2n+3)!} = \frac{1}{(2n+3)(2n+2)}.$$

Since we got a limit of zero, the ratio test implies that our given series converges.

Example 3. For

$$\sum_{n=1}^{\infty} n2^n,$$

the ratio test gives

$$\lim_{n\to\infty} \frac{(n+1)2^{n+1}}{n2^n} = \lim_{n\to\infty} \frac{2(n+1)}{n} = 2.$$

Since this is larger than 1, the given series diverges. (In this case, the n-th term test would also tell us it diverges, since $\lim_{n\to\infty} n2^n \neq 0$.)

Absolute convergence. As stated above, the actual conclusion of the ratio test in the $L < 1$ case is that the given series converges *absolutely*. To say that $\sum a_n$ is *absolutely convergent* means that the series obtained by taking absolute values $\sum |a_n|$ converges. This always implies that the original series converges as well, so absolute convergence is a special type of convergence. If a series converges but does not converge absolutely (so $\sum a_n$ converges but $\sum |a_n|$ does not), we say it is *conditionally* convergent. For instance, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges, as we'll see using the alternating series test we'll discuss next time, but the series of absolute values

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. Thus $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

Next time we'll mention why people care about this distinction between absolute vs conditional convergence. This won't play a big role in this course, so for us the key takeaway is that absolute convergence implies ordinary convergence, which is really all we care about.

Lecture 9: Alternating Series Test

Warm-Up 1. We determine whether or not

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

converges. We compute:

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} = \lim_{n\to\infty} \frac{(n+1)!100^n}{100^{n+1}n!} = \lim_{n\to\infty} \frac{n+1}{100} = \infty,$$

where we use the fact that

$$\frac{(n+1)!}{n!} = \frac{(n+1)n!}{n!} = n+1.$$

Since this limit is ∞ (which we consider to be in the $L > 1$ case), the ratio test tells us that this series diverges.

Warm-Up 2. We determine the values of x for which the series

$$\sum_{n=1}^{\infty} n^2 3^n x^n$$

converges. We will use the ratio test, so first we compute the limit $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ where $a_n = n^2 3^n x^n$. We have:

$$\lim_{n\to\infty} \frac{|(n+1)^2 3^{n+1} x^{n+1}|}{|n^2 3^n x^n|} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) 3|x| = 3|x| \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 3|x|.$$

If this limit is less than 1, the ratio test says that the series converges; while if this limit is greater than 1, the series diverges. So, the series converges at least when $3|x| < 1$, which gives $-\frac{1}{3} < x < \frac{1}{3}$, and diverges at least when $3|x| > 1$, which gives $x < -\frac{1}{3}$ and $\frac{1}{3} < x$.

Now, when this limit is exactly equal to 1, the ratio test is inconclusive. This happens when $3|x| = 1$, so when $x = -\frac{1}{3}$ and $x = \frac{1}{3}$. In these cases we have to go back to the original series: when $x = -1$ the series in question is

$$\sum_{n=1}^{\infty} n^2 3^n (-1)^n$$

and when $x = 1$ the series is

$$\sum_{n=1}^{\infty} n^2 3^n.$$

Both of these diverge by the n-term test since neither $\lim_{n\to\infty} n^2 3^n (-1)^n$ nor $\lim_{n\to\infty} n^2 3^n$ equal 0. Thus, to summarize, the given series only converges for $-\frac{1}{3} < x < \frac{1}{3}$.

Absolute vs conditional. Now we give the reason why the distinction between absolute vs conditional convergence matters. This is NOT something we'll focus on in this course and is more of a "fun fact" everyone should hear about once in their lifetime. Here is the basic idea: rearranging the terms of an absolutely convergent series does not affect the convergence nor the value, whereas rearranging the terms of a conditionally convergent series *could* affect the value!

If we have an infinite sum:

$$a_1 + a_2 + a_3 + a_4 + \cdots,$$

we can rearrange the terms in some way:

$$a_{100} + a_3 + a_{10} + a_4 + a_1 + a_{567} + \cdots \text{ and so on.}$$

The fact is that this new sum does NOT necessarily have the same value as the original one; this is only guaranteed to happen when the original series is absolutely convergent! So, for instance, we've seen previously claimed that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

is conditionally convergent, as we'll soon see. Because of this, rearranging terms could definitely have an effect on the actual value of the series. Even worse, I claim that there is a rearrangement which gives the value π as a result, there is another rearrangement which gives the value e, another giving the value $2^{\sin 1}$, etc: given any real number whatsoever, there is a way to rearrange the terms of a conditionally convergent series to obtain a series whose value is that chosen real number! This cannot happen for absolutely convergent series, where rearrangements affect nothing.

These facts might seem counterintuitive, since rearranging the terms of a finite sum never affects the value:

$$x + y + z + w \text{ is the same as } x + w + z + y \text{ is the same as } z + w + x + y$$

and so on. This is yet another subtle distinction between infinite and finite sums which shows that we have to be careful applying whatever intuition we have for finite sums to infinite sums. Understanding why infinite sums have the properties listed above is way beyond the scope of this course, and, as I said, this will not play a further role for us. But, it is an interesting observation nonetheless!

Example where ratio test doesn't work. Now consider the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 4}.$$

Applying the ratio test gives:

$$\lim_{n\to\infty} \frac{|(-1)^{(n+1)-1}\frac{n+1}{(n+1)^2+4}|}{|(-1)^n\frac{n}{n^2+4}|} = \lim_{n\to\infty} \frac{(n+1)(n^2+4)}{n(n+1)^2+4} = \lim_{n\to\infty} \frac{n^3+n^2+4n+4}{n^3+2n^2+n+4} = 1.$$

Since we got a limit of 1, the ratio test gives us no information.

However, we can instead show that this series converges using the alternating series test, which we will now describe. Since it does converge, we can ask whether it converges absolutely. For this we consider the series obtained by taking absolute values is:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4}.$$

You can show that this diverges by comparing it to $\sum_{n=1}^{\infty} \frac{1}{n}$ using either the direct comparison or limit comparison test. So, since our original series converges but the series of absolute values does not, our original series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$$

is conditionally convergent.

Alternating series. An *alternating series* is a series where the terms alternate between being positive and negative, or negative and positive. For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

is an alternating series. There is nice, simple test for convergence of such series: the *alternating series test*.

In general, an alternating series can be written as

$$\sum_{n=1}^{\infty} (-1)^n b_n \text{ where the } b_n \text{ are positive.}$$

In other words, b_n is what you get when you factor out -1 from the negative terms and keep the positive terms as they are. In the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

we have $b_n = \frac{1}{n}$. (The alternating series test also applies to something like $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, meaning that whether we have $(-1)^n$ or $(-1)^{n-1}$ or something else like $(-1)^{n+4}$ is not important; all that matters is that we have signs which alternate between positive/negative or negative/positive.) The alternating series test says that:

if b_n is decreasing and $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

Thus, for an alternating series, we can demonstrate convergence simply by showing that the terms of the series approach 0 and (after we forget any negative signs) are decreasing, which means that each term is smaller than the one which came before.

Example. We apply the alternating series test to the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

As we said above, in this case we consider $b_n = \frac{1}{n}$. These terms are definitely decreasing since the denominators get larger as n increases (we can also see they decreasing by showing that $f(x) = \frac{1}{x}$ has negative derivative for $x \geq 1$) and since

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

the alternating series test shows that our given series converges.

Fact. The actual value of the series above is:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2.$$

This is not important for the midterm, but is actually something we'll be able to determine once we talk about *power series*. Nonetheless, without knowing that the value is actually $-\ln 2$, we can ask if there is a way in which we can *approximate* the correct value. We'll touch on this a bit next time, and this will play a larger role towards the end of the quarter when we talk about using power series to approximate functions.

Another example. We show that the alternating series

$$\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+4}$$

converges, as we claimed above. In the notation of the alternating series test, here we have

$$b_n = \frac{n}{n^2+4}.$$

First, the function $f(x) = \frac{x}{x^2+4}$ is decreasing since its derivative

$$f'(x) = \frac{(x^2+4) - x(2x)}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$$

is negative for $x > 2$. Thus $b_n = \frac{n}{n^2+1}$ is decreasing. Next,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0,$$

so the alternating series test says that $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$ does converge.

Lecture 10: Convergence Strategies

Warm-Up. We show that
$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{e^n}$$
converges using the alternating series test. The sequence $\frac{n}{e^n}$ consists of all nonnegative terms. Also, the function $f(x) = \frac{x}{e^x}$ is decreasing since its derivative is nonpositive:
$$f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{(1-x)e^x}{e^{2x}} \leq 0 \text{ for } x \geq 1.$$
Hence the sequence $f(n) = \frac{n}{e^n}$ is also decreasing. Finally, we have
$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$
by L'Hopital's rule, so $\lim_{n\to\infty} \frac{n}{e^n} = 0$ as well. We conclude that this series converges by the alternating series test.

In fact, other tests are also applicable in this example. For instance, we can use the ratio test:
$$\lim_{n\to\infty} \frac{|(-1)^{n+1}\frac{n+1}{e^{n+1}}|}{|(-1)^n \frac{n}{e^n}|} = \lim_{n\to\infty} \frac{n+1}{ne} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{e} = \frac{1}{e}.$$
This is limit is smaller than 1, the given series converges absolutely. Also, we can apply the integral test, not to the given series since it does not consist of all positive terms, but instead to the series of absolute values:
$$\sum_{n=1}^{\infty} \frac{n}{e^n}.$$
The integral test will show that this series converges, and hence so does the original series. Again, the point is that multiple convergence tests might be applicable to a given problem, and in a bit we'll discuss strategies for deciding which to apply in which scenario.

Approximating values. Alternating series give us our first example of a series where it is possible to come up with good approximations to the value of a series in cases where we can't determine the actual value. Say that $\sum_{n=1}^{\infty}(-1)^n b_n$ denotes an alternating series. Recall that k-th partial sum of this series is the sum of the first however many terms up to $n = k$:
$$-b_1 + b_2 - b_3 + \cdots + (-1)^k b_k.$$
The idea is that as we get partial sums with more and more terms, such partial sums should be giving better and better approximations to the actual value of the series in question. In particular, the expression
$$|(\text{actual value}) - (k\text{-th partial sum})|,$$
so absolute value of the difference between the actual value and a partial sum approximation, is precisely telling us how good of an approximation the k-th partial sum is to the actual value. Our goal is to be able to "control" how bad this "error' term can be.

For an alternating series, the fact is that this error term can be bounded by one of the b_n terms itself. To be clear, the fact is that
$$\left|(\text{actual value of } \sum(-1)^n b_n) - (k\text{-th partial sum})\right| \leq b_{k+1}.$$

So, the point is that the "error" in approximating the actual value of $\sum (-1)^n b_n$ with the partial sum

$$-b_1 + b_2 - b_3 + \cdots + (-1)^k b_k$$

is no more than the value of b_{k+1}. The smaller this "error" is, the better an approximation we have. The intuition for this fact comes from the following. The actual value of an alternating series is an infnite sum

$$b_0 - b_1 + b_2 - b_3 + \cdots - b_k + b_{k+1} - b_{k+2} + \cdots$$

and the k-th partial sum is something like

$$b_0 - b_1 + b_2 - b_3 + \cdots - b_k.$$

The difference between these is the portion of the infinite sum starting with the b_{k+1} term:

$$b_{k+1} - b_{k+2} + \cdots ,$$

and these difference is smaller than b_{k+1} itself since this difference is b_{k+1} minus something smaller, plus something smaller still, minus something even smaller, and so on. (Here we use the fact that the b_n are positive and decreasing towards zero.)

As mentioned previously, having such a bound on the "error" gives a way to determine how good our approximations actually are. This is tough to come up with for series in general, although we'll see a way to proceed with *Taylor series* later on, where this idea will truly shine.

Example. Going back to a series we saw last time:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

in this case we get that the (absolute value of the) difference between the actual value (which we claimed last time was $-\ln 2$) and the value of the k-th partial sum is bounded by:

$$|(\text{actual value}) - (k\text{-th partial sum})| \leq \frac{1}{k+1}.$$

For instance, looking at the 9-th partial sum gives:

$$\left|(\text{actual value}) - \left(-1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{8} - \frac{1}{9}\right)\right| \leq \frac{1}{10}.$$

So, the error between the actual value of $-\ln 2$ and the approximation given by

$$-1 + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{9}$$

is no more than $\frac{1}{10} = 0.1$. This partial sum is (if you work it out) roughly -0.7456, so the error between this and the actual value is

$$|-\ln 2 - (-0.7456)| \approx 0.005,$$

which is indeed less than 0.1.

Another example. Consider the series from the Warm-Up:

$$\sum_{n=1} (-1)^n \frac{n}{e^n}$$

Say we want to approximate the actual value of this sum to within an error of $\frac{1}{100}$. The difference between the k-th partial sum of this series and the actual value of the series is bounded by the following:

$$|(k\text{-th partial sum}) - (\text{actual value})| \leq \frac{k+1}{e^{k+1}}$$

since $\frac{k+1}{e^{k+1}}$ is the term which would occur next in the series after the final term $\frac{k}{e^k}$ of the k-partial sum. Thus in order to guarantee that our approximation using a partial sum is within $\frac{1}{100}$ of the actual sum, we should look for k satisfying

$$\frac{k+1}{e^{k+1}} \leq \frac{1}{100}.$$

We can check that $k = 6$ satisfies this inequality, so that the k-partial sum is what we want. That is, the sum

$$-\frac{1}{e} + \frac{2}{e^2} - \frac{3}{e^3} + \frac{4}{e^4} - \frac{5}{e^5} + \frac{6}{e^6}$$

gives the actual value of this series to within an error of $\frac{1}{100}$.

Strategies for testing convergence. We've now seen multiple ways to see if a series converges. It can seem daunting to keep all of these in mind, and also to get comfortable with deciding which to use in a given scenario. But, this is something which comes more easily with practice and picking up on certain patterns. (Indeed, pattern recognition is a good skill to develop in general, not just for math courses.) We finish by giving a brief summary of strategies to use.

We'll list the various convergence tests in the order in which they should be applied, from simpler to more challenging. We'll also highlight specific things to look for which might suggest one test in particular is the right one to use. Here we go:

- n-th term test: As a first step, take 10-20 seconds to see if the n-th term test applies. If $\lim_{n\to\infty} a_n \neq 0$, then you are done—the series $\sum a_n$ diverges for sure. No use in going through some elaborate test if this already works. For series where this is the way to go, it should be quick to determine if it does indeed apply, so only spend 10-20 seconds max.

- geometric series, p-series test: If your series is a geometric series or one of the form $\sum \frac{1}{n^p}$, then again it is quick to tell what happens. To be sure, for the geometric case, you should see if you can write the terms in your series as constant(constant)^n.

- alternating series test: If your series is alternating, then this is a good thing to try. (Not the only thing: the ratio test might also be good.) Having a $(-1)^n$ term or something similar is a tip that you might think about applying this test.

- ratio test: In most cases where one of the above does not apply, the ratio test should be your first thought. This is especially true for series which involve factorials, or series terms having n as the exponent which are *not* geometric series, which is the case when you have a non-constant multiplied by something which has an exponent of n.

I'm grouping these first four tests together as being more "direct", in the sense that they don't require coming up with something else to compare a given series to, nor do they require having to compute a whole separate integral. Each of these tests, when they apply, gives us the answer right away without any additional work.

The final tests are ones which usually require more work since they are not as straightforward as the above, but in certain cases they might be the right thing to use:

- limit comparison test: This is the next thing to try by default, since it is usually simpler to work through than the final two tests. In particular, for series with fractions whose numerator and denominators only involve powers of n, this should be your go-to-test.

- integral test: If nothing has worked so far, try the integral test next. In particular, if you can see that the function $f(x)$ describing the terms of your series $\sum f(n)$ is quick to integrate, maybe try this first. But of course, be sure to check the hypothesis of the integral test to make sure it applies: positive, continuous, decreasing function.

- direct comparison test: And finally, we try the direct comparison test. This is probably the toughest test to apply, since it not only requires a series to compare to (as does the limit comparison test), but it also requires coming up with some inequalities, which is usually more work. However, for any series which involves sine or cosine, perhaps this is the test you might think of applying before the others, since sine and cosine terms are usually simple to bound.

Lecture 11: Power Series

Warm-Up. Which test to use? ***TO BE FINISHED***

Series as functions. Let us go back to the basic geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ where } |x| < 1.$$

We now focus on the idea that we can view the right side as a *function* by treating x as a variable: plugging in a value of x into this series gives the value on the left side, so in other words gives the same value as the function $f(x) = \frac{1}{1-x}$. We say that the series $\sum_{n=0}^{\infty} x^n$ *represents* the function $\frac{1}{1-x}$ on the interval $(-1, 1)$, which in this case is the interval characterizing the values x for which the series converges, and hence for which the equality

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

is valid. Representing functions as series is the whole reason why we care about series in the first place, since, as we'll see, this will gives us new ways of studying functions, which will be especially useful for functions which are otherwise difficult to understand.

Consider now the series

$$\sum_{n=0}^{\infty} 3^n x^n$$

where again we treat x as a variable. By writing this as $\sum_{n=0}^{\infty} (3x)^n$, we see that this too is geometric since we can obtain it from

$$\sum_{n=0}^{\infty} y^n$$

by setting $y = 3x$. (We are using y as the variable in this latter series to avoid confusing it with the x in the previous series.) Making the substitution $y = 3x$ in

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

results in

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$$

so we would say that the series on the right is a series representation of the function $\frac{1}{1-3x}$. But again we should be careful about clarifying the values of x for which this equality is actually valid. Since this was derived from the geometric series $\sum y^n$ by setting $y = 3x$, and this geometric series only converges for $|y| < 1$, this derived series converges for $|3x| < 1$, or equivalently for $|x| < \frac{1}{3}$. Thus we get that

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n \text{ is valid for } x \text{ in } (-\tfrac{1}{3}, \tfrac{1}{3}).$$

Similarly, by setting $y = -x$ in the geometric series above we find that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

is a valid series representation of the function $\frac{1}{1+x}$ on the interval $(-1, 1)$. Looking ahead to the types of things we will soon be doing, we could now ask about the *derivative* of $\frac{1}{1+x}$: this derivative is $-\frac{1}{(1+x)^2}$, and so if we could in turn differentiate the *series* $\sum_{n=0}^{\infty}(-1)^n x^n$, we could obtain a series representation of $-\frac{1}{(1+x)^2}$:

$$-\frac{1}{(1+x)^2} = \text{derivative of } \sum_{n=0}^{\infty} (-1)^n x^n.$$

Just as well, if we *integrate* $\frac{1}{1+x}$ to get $\ln|1+x|$, we could ask about doing the same thing to the series representation of $\frac{1}{1+x}$ in order to obtain a power series representation of $\ln|1+x|$:

$$\ln|1+x| = \text{integral of } \sum_{n=0}^{\infty} (-1)^n x^n.$$

We will discuss what it means to differentiate and integrate series soon enough.

Power series. All of the series we used above to representation certain functions are examples of *power series*, which are a type of series involving powers of a variable x. To be clear, a power series is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where the c_n are numbers, x is a variable, and a is a number we call the "center" of the series. (We say that this is a power series *centered* at a.) The idea is that, because x is a variable, we view a power series as defining a function depending on x, and that the functions defined by power series often turn out to be functions we all know and love.

A power series should be viewed as a type of "infinite" polynomial, since it looks like a polynomial only with a possibly infinite number of terms:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

More precisely, the partial sums of this series are literal polynomials, and so the point is that we will be able to approximate the function to which the power series converges by polynomial expressions.

Example 1. Consider the power series

$$\sum_{n=0}^{\infty}(3x-1)^n.$$

First, we should be clear about why this is actually a power series, since we defined a power series to be one involving powers of $x-a$ for some center a (with the coefficient of x being 1), but here we have powers of $3x-1$, where the coefficient of x is 3. The point is that we can rewrite this series to put it into the correct form:

$$\sum_{n=0}^{\infty}(3x-1)^n = \sum_{n=0}^{\infty}\left[3\left(x-\tfrac{1}{3}\right)\right]^n = \sum_{n=0}^{\infty}3^n\left(x-\tfrac{1}{3}\right)^n.$$

(To be clear, we wrote $3x-1$ as $3(x-\frac{1}{3})$ by factoring out the 3.) This makes the power series form clear, and we see that this is indeed a power series centered at $\frac{1}{3}$; $3x-1$ might seem to suggest that the center is 1, but we do not know the center for sure until we have it in the correct $x-a$ form.

Now, we can obtain this given series from the geometric series

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

by setting $y = 3x-1$:

$$\frac{1}{1-(3x-1)} = \sum_{n=0}^{\infty}(3x-1)^n.$$

This series will converge when $|y| < 1$, so when

$$|3x-1| < 1.$$

Taking into account that this should be a power series centered at $\frac{1}{3}$, we write this inequality as

$$3\left|x-\tfrac{1}{3}\right| < 1, \text{ or } \left|x-\tfrac{1}{3}\right| < \tfrac{1}{3}.$$

Thus the given series converges for x in $(0, \frac{2}{3})$, and so we have that

$$\frac{1}{2-3x} = \sum_{n=0}^{\infty}(3x-1)^n = \sum_{n=0}^{\infty}3^n\left(x-\tfrac{1}{3}\right)^n$$

represents the function $\frac{1}{2-3x}$ as a power series centered at $\frac{1}{3}$ on the interval $(0, \frac{2}{3})$.

Example 2. Of course, not all power series can be derived from a clever manipulation of a geometric series. For instance, consider the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We will soon see that this series actually converges for all x (we will use the ratio test to verify this), and that it actually converges to the function e^x! (That's an exclamation mark, not a factorial.)

If we take this as given for now, then we can derive a whole bunch of other power series representations. Take

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

as our starting point, where again we use y as the variable to distinguish it from the x which will appear after we make a substitution. Setting $y = 2x$ gives

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n,$$

which is thus a power series representation of the function e^{2x}. (The coefficients c_n of this power series are $\frac{2^n}{n!}$.) Instead if we set $y = x + 3$ above we get

$$e^{x+3} = \sum_{n=0}^{\infty} \frac{(x+3)^n}{n!}$$

which express the function e^{x+3} as a power series centered at -3. (The center is -3 and not 3, since we must write $x + 3$ as $x - (-3)$ in order to make it look like the $x - a$ term required in the power series format.) So, the upshot is that once we know a few power series representations, we get a whole lot more by considering substitutions, or later derivatives and integrals.

Lecture 12: Interval of Convergence

Warm-Up 1. ***TO BE FINISHED***

Warm-Up 2. ***TO BE FINISHED***

Interval of convergence. So far we know that the power series $\sum x^n$ converges for x in $(-1, 1)$ based on what we know about geometric serires, and that the series $\sum \frac{x^n}{n!}$ converges for all x, because I said so last time. But how do we determine the values of x for which a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

converges in general? The key fact is that any power series converges for x in some *interval* around its center a, which could have zero length, positive length, or infinite length. This interval is called the *interval of convergence* of the power series, and half its length—i.e. the distance from the center to either endpoint—is called its *radius of convergence*.

Find the interval and radius of convergence is an application of the ratio test, as the following examples will make clear. It is on this interval that we can make sense of saying that the power series in question represents a *function*, since this interval characterizes the values of x we can actually plug into that function and have the resulting value make sense.

Example 1. Consider the power series

$$\sum_{n=1}^{\infty} n(x-2)^n.$$

We will determine when this converges using the ratio test. If we set $a_n = n(x-2)^n$, the limit in the ratio test becomes:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)|x-2|^{n+1}}{n|x-2|^n} = \lim_{n\to\infty} \frac{(n+1)|x-2|}{n} = |x-2| \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = |x-2|.$$

Thus according to the ratio test, the given series converges for sure when this resulting limit value is smaller than 1, so when $|x-2| < 1$, and diverges for sure when $|x-2| > 1$. The inequality $|x-2| < 1$ is the same as

$$-1 < x-2 < 1, \text{ or equivalently } 1 < x < 3.$$

Hence, the given power series converges at least for x in the interval $(1,3)$. The radius of convergence, the distance from the center 2 of this series to the endpoints of this interval, in this case is 1. Note that this interval can be obtained simply by subtracting and adding the center to the radius in order to get the endpoints: $(2-1, 2+1) = (1,3)$.

But, we cannot just yet say that the interval of convergence is $(1,3)$. Recall that the ratio test is inconclusive when $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$. In our case, this means that the ratio test is inconclusive when $|x-2| = 1$, so when $x = 1$ or $x = 3$. The point is that this method of finding the radius and interval of convergence will say nothing about the endpoints of the resulting interval, and so we have to see what happens at those endpoints separately. When $x = 1$ our given series becomes

$$\sum_{n=1}^{\infty} n(-1)^n,$$

which diverges since $\lim_{n\to\infty} n(-1)^n \neq 0$, and when $x = 3$ our series is

$$\sum_{n=1}^{\infty} n,$$

which also diverges. This means that neither 1 nor 3 should be included in the interval of convergence, so that the interval of convergence is indeed $(1,3)$.

In general, the ratio test will give us most of the interval of convergence $(a-R, a+R)$, where a is the center and R is the radius, but since the ratio test is inconclusive at x equals either endpoint, we have to check what happens at these separately.

Example 2. We determine the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n}.$$

Using the ratio test with $a_n = \frac{2^n x^n}{n}$, we get:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{2^{n+1}|x|^{n+1}}{n+1}}{\frac{2^n|x|^n}{n}} = \lim_{n\to\infty} \frac{2|x|n}{n+1} = 2|x| \lim_{n\to\infty} \frac{n}{n+1} = 2|x|,$$

where we can find that the limit of $\frac{n}{n+1}$ is 1 by dividing numerator and denominator by n. Thus, by the ratio test, our power series converges at least for x satisfying

$$2|x| < 1, \text{ or equivalently } |x| < \frac{1}{2}.$$

Hence this series has radius of convergence $\frac{1}{2}$; in general this comes from how large $|x - \text{center}|$ can be, so the goal is to turn the ratio test inequality into one of the form $|x - \text{center}| < R$, where R is then the radius. The interval of convergence is thus at least $(0 - \frac{1}{2}, 0 + \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2})$.

But now we see what happens at the endpoints. When $x = 1$ we get the series

$$\sum_{n=0} \frac{2^n \frac{1}{2^n}}{n} = \sum_{n=0}^{\infty} \frac{1}{n},$$

which diverges by the p-series test, but when $x = -1$ we get

$$\sum_{n=0} \frac{2^n(-\frac{1}{2})^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n},$$

which actually converges by the alternating series test. Thus, for the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n},$$

$x = -1$ should be included in the interval of convergence, so the full interval of convergence is the half-closed/half-open interval $[-1, 1)$.

Example 3. Now we look at the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which we have previously claimed converges for all values of x. Setting $a_n = \frac{x^n}{n!}$, the ratio test tells us that this series will converge when

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1.$$

In our case we thus consider:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Because this is smaller than 1 no matter what x is, we thus conclude that series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

does converges for all x. The interval of convergence is thus $(-\infty, \infty)$, and we say that the radius of convergence is ∞. In this case there are no endpoints to check.

Example 4. Next we determine the radius and interval of convergence of

$$\sum_{n=0}^{\infty}(2n+1)!(x-1)^n.$$

Here, $\lim \frac{|a_{n+1}|}{|a_n|}$ becomes:

$$\lim_{n\to\infty} \frac{(2(n+1)+1)!|x-1|^{n+1}}{(2n+1)!|x-1|^n} = \lim_{n\to\infty} \frac{(2n+3)!}{(2n+1)!}|x-1| = \lim_{n\to\infty}(2n+3)(2n+2)|x-1|.$$

Now we have to be careful: the $(2n+3)(2n+2)$ portion goes to ∞, but whether we actually get an infinite value for the limit will depend on whether the $|x-1|$ term is zero. Indeed, if $x = 1$, $|x-1| = 0$ and this limit is:

$$\lim_{n\to\infty}(2n+3)(2n+2)0 = \lim_{n\to\infty} 0 = 0,$$

and since this is less than 1 the ratio test tells us that the given series *does* converge when $x = 1$. However, if $x \neq 1$, $|x-1| \neq 0$ and so in this case

$$\lim_{n\to\infty}(2n+3)(2n+2)|x-1| = \infty,$$

and hence the series does not converge in this case.

Thus this series converges only when $x = 1$, so we say that the interval of convergence (which isn't really an interval in this case) consists of just the single point 1, and so the radius of convergence is 0. (This makes sense since a single point has zero length, and half of zero is zero.)

Example 5. Finally, we determine the radius and interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^n(x-2)^n}{2^n(n+1)}.$$

Note that this is a power series centered at 2. We compute:

$$\lim_{n\to\infty} \frac{|x-2|^{n+1}}{2^{n+1}(n+2)} \frac{2^n(n+1)}{|x-2|^n} = \lim_{n\to\infty} \frac{|x-2|}{2}\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty} \frac{|x-2|}{2} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = \frac{|x-2|}{2}.$$

By the ratio test this series converges when

$$\frac{|x-2|}{2} < 1, \text{ so when } |x-2| < 2.$$

The the radius of convergence is 2 and the interval of convergence is at least $(2-2, 2+2) = (0, 4)$.

Now we check for convergence at the endpoints. For $x = 0$ our series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n(-2)^n}{2^n(n+1)} = \sum_{n=2}^{\infty} \frac{1}{n+1},$$

which diverges as we can see doing a limit comparison test with the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$. Thus 0 is not in the interval of convergence. For $x = 4$ this series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{2^n(n+1)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1},$$

which converges by the alternating series test. (Note that here I'm not working out the details of the alternating series test, but this is something you should be able to do.) Thus 4 is in the interval of convergence, so the interval of convergence of the given series is $(0, 4]$.

Lecture 13: Series Manipulations

Warm-Up 1. We find the interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2x-5)^n}{n^2 2^{n+1}}.$$

TO BE FINISHED

The various examples we've seen illustrate the different things which can happen: either the radius of convergence is 0 in which case the series converges only at its center, or the radius of convergence is infinite in which case the interval of convergence is $(-\infty, \infty)$, or the radius of convergence is some positive number R in which case the interval of convergence looks like $(a - R, a + R)$ (where a is the center) and possibly includes none, one, or both of the endpoints.

Warm-Up 2. We find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n!)}.$$

TO BE FINISHED

Series substitutions. We now come back to the idea of representing various functions as power series. For instance, we can ask: what function does the power series

$$\sum_{n=0}^{\infty} (-1)^n x^{2n}$$

represent? This is something we can determine by manipulating the series

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

along the lines of some examples we saw before. In this case, setting $y = -x^2$ in this series gives

$$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1) x^{2n},$$

which is precisely the series we're asking about. Since

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

we thus get that

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

so the power series we had at the beginning represents the function

$$\frac{1}{1+x^2}.$$

Moreover, we can determine that this power series has interval of convergence $(-1,1)$—either by using the ratio test and checking endpoints, or by using the convergence for $|y|<1$ of the geometric series with $y=-x^2$ from which this was derived—so the power series representation

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

is valid on the interval $(-1,1)$.

The overall idea in this example is what we are making a *substitution* into one series in order to obtain another: substitute $-x^2$ in place of x in $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. We just used y above in order to make this substitution clear.

Example 1. As another example using substitution, again start with

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \text{ for } |y|<1.$$

Making the substitution $y = 1-x$ gives

$$\frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n \text{ for } |1-x|<1,$$

which we can write as

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \text{ for } |x-1|<1.$$

This thus gives a representation of the function $\frac{1}{x}$ as a power series centered at 1, which is valid on the interval $(1-1, 1+1) = (0,2)$. Of course, this interval of convergence can also be found by using the ratio test to find the radius of convergence, which is 1.

Example 2. Starting with

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which is an equality we still have to justify, substituting x^2 in place of x gives

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Instead, substituting $-x$ in place of x in the series for e^x above gives

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

Both of these series representations turn out to converge for all x, since the series which are used have interval of convergence $(-\infty, \infty)$.

Now we can go further. For instance, taking the series representation for e^{x^2} above and multiplying through by x gives a series representation of xe^{x^2}:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \rightsquigarrow xe^{x^2} = x\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!},$$

which is valid for all x. If instead we want to determine the function represented by the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{n!},$$

we note that this can be written as

$$\sum_{n=0}^{\infty} \frac{(-x^3)^n x^2}{n!} = x^2 \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!}.$$

Since the final series is obtained by replace x by $-x^3$ in $e^x = \sum_{n=0} \frac{x^n}{n!}$, we find that our series equals the function

$$x^2 e^{-x^3} = x^2 \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{n!}.$$

Differentiating series. Manipulating a power series by making a substitution (such as $y = -x^2$ or $y = 1 - x$) like we did above is one way of manipulating one series in order to produce another. Another way of manipulating a series comes from differentiation.

Write a power series as an infinite sum:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

We can differentiate this just as we're used to, by differentiating each term one at a time:

$$\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots .$$

This results in the formula:

$$\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = \sum_{n=0}^{\infty} c_n n (x-a)^{n-1},$$

where the $n(x-a)^{n-1}$ piece comes from differentiating $(x-a)^n$. Note that the $n = 0$ term in the resulting series is 0 (because plugging in $n = 0$ into $c_n n(x-a)^{n-1}$ gives zero), so we can rewrite the series to start at $n = 1$ (the first nonzero term) instead:

$$\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1}.$$

This is just reflecting the fact that the constant c_0 term in

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

disappears after taking derivatives. This process is called *term-by-term differentiation*.

Example 4. Differentiating the standard geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

gives

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)\\ \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} \frac{d}{dx}(x^n)\\ &= \sum_{n=1}^{\infty} nx^{n-1}.\end{aligned}$$

To be clear, we start the final series at $n = 1$ since the $n = 0$ term is zero anyway, so it is not worth writing. Multiplying through by x would then give the following series representation:

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n.$$

Example 5. Consider the function $f(x)$ defined by the following series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which is know is defined for all x. The term-by-term derivative is:

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}.$$

Now, the resulting series can be written to start at $n = 0$ instead of $n = 1$, which will have the effect of increasing every n which appears within the sum by 1:

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To be clear, the series which starts at $n = 1$ has terms which look like:

$$\frac{x^{1-1}}{(1-1)!} + \frac{x^{2-1}}{(2-1)!} + \frac{x^{3-1}}{(3-1)!} + \cdots$$

while the series which starts at $n = 0$ looks like:

$$\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots,$$

which is the same thing. That is, reindexing does not actually change the value of sum, only how it is written. In general, dropping the index from $n = k$ to $n = 0$ corresponds to replacing n by $n + k$ in the terms of the series.

And so the point is that the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ equals its own derivative, which gives evidence to the claim that this series equals e^x, which we will come back to later.

Example 6. Finally, consider the series

$$\sum_{n=1}^{\infty} (-1)^n 2nx^{2n-1}.$$

The key observation here is that this series is precisely the result of differentiating the series

$$\sum_{n=0}^{\infty} (-1)^n x^{2n}$$

we saw earlier. Indeed, differentiating term-by-term gives

$$\left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right)' = \sum_{n=1}^{\infty} (-1)^n 2nx^{2n-1}$$

since $2nx^{2n-1}$ is the derivative of x^{2n}. Since

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

the series we're looking at in this example is should represent the function obtained by differentiating

$$\frac{1}{1+x^2}.$$

In other words, taking derivatives of both sides of

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

gives

$$\left(\frac{1}{1+x^2} \right)' = \sum_{n=1}^{\infty} (-1)^n 2nx^{2n-1},$$

so the given series in this example represents the function

$$\frac{-2x}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^n 2nx^{2n-1}.$$

Furthermore, if we divide by the extra factor of 2 which appears, we get the representation

$$-\frac{x}{(1+x^2)^2} = \sum_{n=1}^{\infty} (-1)^n nx^{2n-1}$$

Lecture 14: More on Manipulations

Warm-Up 1. We determine the function which is represented by the series

$$\sum_{n=2}^{\infty} n(n-1)x^n.$$

The key observation is that this series is *almost* what we get if we take *two* derivatives of

$$\sum_{n=0}^{\infty} x^n.$$

Indeed, taking one derivative gives

$$\sum_{n=1}^{\infty} nx^{n-1},$$

and taking another gives

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2}.$$

(Note that here we've written this series to start at $n=2$; we could have written it to start at $n=0$ instead only that the $n=0$ term itself would be 0 because of the n coefficient, and the $n=1$ term would also be zero because of the $n-1$ coefficient. The first nonzero term is the $n=2$ term, which is why we write the series to start at this value.) Since the original series $\sum x^n$ represented the function $\frac{1}{1-x}$, the series after we take two derivatives will represent

$$\left(\frac{1}{1-x}\right)'' = \frac{2}{(1-x)^2}.$$

Thus so far we have that

$$\frac{2}{(1-x)^2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}.$$

The only difference between this series and the one we want is the power of x, but that can be fixed by multiplying through by x^2:

$$\frac{2x^2}{(1-x)^2} = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^2x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^n.$$

Thus we conclude that the original series at the start of this Warm-Up is a series representation of the function

$$\frac{2x^2}{(1-x)^2}.$$

Warm-Up 2. ***TO BE FINISHED***

Integrating series. Similarly, we can integrate one power series in order to produce another. Integrating

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

should give

$$\int \left(\sum_{n=0}^{\infty} c_n (x-a)^n \right) dx = c_0 x + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + c_3 \frac{(x-a)^4}{4} + \cdots ,$$

so

$$\int \left(\sum_{n=0}^{\infty} c_n (x-a)^n \right) dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

To be clear, the

$$\frac{(x-a)^{n+1}}{n+1}$$

term comes from integrating $(x-a)^n$. When considering indefinite integrals we should also throw on a $+C$ term at the end as usual. This process is called *term-by-term integration*.

Example. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}.$$

Note that if we start with

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n,$$

integrating both sides gives

$$-\ln|1-y| = \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} + C.$$

The point is that integration is the type of operation which can give additional n terms (in this case $n+1$) in the denominator of a series expression. The unknown constant of integration C can be found by plugging in $y=0$ into both sides: this gives

$$-\ln 1 = \sum_{n=0}^{\infty} 0 + C, \text{ or } 0 = 0 + C.$$

Hence $C = 0$ so

$$-\ln|1-y| = \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1}.$$

Making the substitution $y = -x$ then gives

$$-\ln|1+x| = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n+1}.$$

Now, this is almost the series we want, only that the series we want starts at $n=1$ and involves

$$\frac{x^n}{n} \text{ instead of } \frac{x^{n+1}}{n+1}.$$

However, note that we can *rewrite* the series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n+1}$$

to make it start at $n = 1$ instead. To make this clear, let us instead use m as the indexing variable:

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{m+1}}{m+1}.$$

If we set $n = m+1$, we get a series starting at $n = 1$ (since $n = 1$ when $m = 0$) which looks like:

$$\sum_{m=0}^{\infty} \frac{(-1)^{m+1} x^{m+1}}{m+1} = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n},$$

and this latter series is the one we want. We conclude that

$$-\ln|1+x| = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

is the function our given series represents.

As in the previous example, we can phrase this the other way around. Say we want to find the series which represents

$$-\ln|1+x|.$$

We note that this function is obtained by integrating

$$\frac{1}{1+x},$$

so that if we know how to represent this latter function as a series, we can integrate to find a representation of the function we want. Since

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n,$$

which comes from setting $y = -x$ in the standard series expression for $\frac{1}{1-y}$, we get after integrating that:

$$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C.$$

The unknown constant C can be found by setting $x = 0$ in this expression, so $0 = 0 + C$ and hence $C = 0$, and thus

$$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

We can reindex this series to start at $n = 1$ instead (this will have effect of replacing each n showing up in the series expression with $n-1$) to get

$$\ln|1+x| = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n},$$

and finally multiplying through by -1 to get

$$-\ln|1+x| = \sum_{n=1}^{\infty} -(-1)^{n-1} \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

as a series expression for $-\ln|1+x|$. This series (if you work it out) has interval of convergence $(-1,1]$, and since 1 is then in this interval, we get the equality

$$-\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

after setting $x = 1$. This was a claim I made previously when discussing alternating series, and now we can see why this is the correct value of this series.

Lecture 15: Taylor Series

Warm-Up 1. ***TO BE FINISHED***

Warm-Up 2. We find a series representation of the function

$$x \ln|1-x^3|.$$

To build up to this, we start by finding a series representation of

$$\ln|1-x^3|.$$

Since

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n,$$

integrating both sides gives

$$-\ln|1-y| = \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} + C,$$

and plugging in $y = 0$ to get that $C = 0$ leaves us with

$$-\ln|1-y| = \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1}.$$

We can multiply through by -1 and rewrite the series on the right to start at $n = 1$ (and replacing n by $n-1$ accordingly) to get

$$\ln|1-y| = \sum_{n=1}^{\infty} -\frac{y^n}{n}.$$

Setting $y = x^3$ gives

$$\ln|1-x^3| = \sum_{n=1}^{\infty} -\frac{x^{3n}}{n},$$

where we used the fact that $(x^3)^n = x^{3n}$, and finally multiplying through by x gives

$$x \ln|1-x^3| = \sum_{n=1}^{\infty} -\frac{x x^{3n}}{n} = \sum_{n=1}^{\infty} -\frac{x^{3n+1}}{n}$$

as the desired representation.

Warm-Up 3. Finally we find a series representation for $\arctan x$. The key fact we need is that the derivative of this functions is $\frac{1}{1+x^2}$:

$$(\arctan x)' = \frac{1}{1+x^2}.$$

Thus if we have a series representation of $\frac{1}{1+x^2}$, integrating term-by-term will give a series representation of $\arctan x$. We've seen before that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

which comes from making the substitution $y = -x^2$ in

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n.$$

Integrating both sides of

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

gives

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C.$$

Setting $x = 0$ gives $0 = 0 + C$, so $C = 0$ and thus

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

is the desired representation.

Determining coefficients. We now come to the problem of trying to represent a function as a series a bit more systematically, in a way which does not depend on having to express our function—either via substitution, differentiation, or integration—in terms of another function whose power series representation is already known.

Suppose we have already expressed $f(x)$ as a power series centered at a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots.$$

Setting $x = a$ in this expression gives

$$f(a) = c_0 + c_1 0 + c_2 0 + c_3 0 + \cdots = c_0,$$

so we first conclude that the unknown constant term c_0 *must* be $f(a)$. Now, taking the derivative of our series expression gives:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots.$$

Setting $x = a$ gives

$$f'(a) = c_1 + 2c_2 0 + 3c_3 0 + \cdots = c_1,$$

so the coefficient c_1 of $x-a$ in our series expression must be $f'(a)$. Taking another derivative gives

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + \cdots$$

and setting $x = a$ gives

$$f''(a) = 2c_2, \text{ so } c_2 = \frac{f''(a)}{2}.$$

Taking another derivative gives

$$f^{(3)}(x) = 2 \cdot 3c_2 + \text{stuff involving } x - a$$

and setting $x = a$ gives

$$f^{(3)}(a) = 2 \cdot 3c_2, \text{ so } c_2 = \frac{f^{(3)}(a)}{3!}.$$

In general, the n-th derivative of

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

looks like

$$f^{(n)}(x) = n!c_n + \text{stuff involving } x - a,$$

and setting $x = a$ gives

$$f^{(n)}(a) = n!c_n + 0, \text{ so } c_n = \frac{f^{(n)}(a)}{n!}.$$

Taylor series. The point of all this is to say that if we want to express f as a power series centered at a:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n,$$

the coefficients needed *must* be given by $c_n = \frac{f^{(n)}(a)}{n!}$; we have no choice! The resulting power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the *Taylor series* of f centered at a and is the *only* power series which could posibly equal the function $f(x)$ on its interval of convergence. So, representing a function as a power series really comes down to finding its Taylor series.

Those Taylor series which are centered at 0 show up often enough that we give them the special name of *Maclaurin series*; so, to be clear, the Maclaurin series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Essentially, Maclaurin series are the most basic (and most important) types of Taylor series.

Example 1. We determine the Maclaurin series (i.e. Taylor series centered at 0) of e^x, and then we determine its Taylor series centered at -3. To compute either one we need to start by determining the derivatives of e^x, which is easy in this case since e^x equals its own derivative:

$$f(x) = e^x, \ f'(x) = e^x, \ f''(x) = e^x, \ f^{(3)}(x) = e^x, \text{ and in general } f^{(n)}(x) = e^x.$$

Evaluating these at 0 gives

$$f^{(n)}(0) = e^0 = 1 \text{ for all } n,$$

so the coefficient of x^n in the Maclaurin series is

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{n!}.$$

Hence the Maclaurin series of e^x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which is the series expression for e^x we mentioned previously. We saw back a few lectures ago that this series has infinite radius and interval of convergence (using the ratio test), so we conclude that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all values of } x.$$

In particular, setting $x = 1$ gives

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots,$$

which is an interesting way of expressing the number e. (In fact, as we'll mention soon enough when we talk about Taylor series approximations, this series is *precisely* how computers and calculators come up with decimal expressions for e.)

Now, the Taylor series for e^x centered at -3 can be computed similarly, only that now we evaluate the derivatives we had above at -3. The coefficient of $(x+3)^n$ (which is $(x - \text{center})^n$) in the Taylor series of e^x centered at -3 is then

$$\frac{f^{(n)}(-3)}{n!} = \frac{e^{-3}}{n!},$$

so the Taylor series centered at -3 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{e^{-3}}{n!}(x+3)^n.$$

This series also has infinite radius of convergence, so

$$e^x = \sum_{n=0}^{\infty} \frac{e^{-3}}{n!}(x+3)^n (\text{which holds for all } x)$$

gives another way of representing e^x as a series, only this time as a power series centered at -3 instead of 0. For instance, setting $x = 1$,

$$e = \sum_{n=0}^{\infty} \frac{e^{-3}}{n!} 4^n$$

gives another way of expressing the number e as an infinite sum.

To note one last thing about this example, here we computed the Taylor series of e^x centered at -3 directly using the definition of a Taylor series, but in this case we can also computed it using the Maclaurin series of e^x. Since e^x can be written as

$$e^x = e^{-3}e^{x+3},$$

we can find a series representation for e^x by taking one for e^{x+3} and then multiplying by e^{-3}. Since

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

based on the Maclaurin series we found above for e^x, setting $y = x + 3$ gives

$$e^{x+3} = \sum_{n=0}^{\infty} \frac{(x+3)^n}{n!},$$

so

$$e^x = e^{-3}e^{x+3} = e^{-3}\sum_{n=0}^{\infty} \frac{1}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{e^{-3}}{n!}(x+3)^n,$$

which is precisely the Taylor series for e^x centered at 3 we found before. This is meant to be yet another example of the use of manipulation to turn one series into another.

Example 2. Consider the indefinite integral

$$\int e^{x^2}\,dx.$$

We've mentioned many times in this course that this is not an integral which can be computed directly, so up until now we've haven't been able to say anything more about what this integral actually is. However, the point is that now we *do* have a way of making sense of this integral, since we can represent it as a series! Since

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!},$$

setting $y = x^2$ gives

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Integrating both sides gives

$$\int e^{x^2}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)} + C,$$

which is our desired series representation.

Integrals like the one here and others which cannot be computed directly using the integration techniques we spend the first portion of the course studying show up all the time in applications, in fact these types of integrals probably show up more often than integrals which *can* be computed directly. The point is that in these applications the best you can do with such integrals is find a way to represent them as power series, which, as we'll talk about later when discussing approximations, is for most purposes good enough.

Example 3. Finally, we compute the Maclaurin series of $\cos x$. We first need derivatives:

$$f(x) = \cos x,\ f'(x) = -\sin x,\ f''(x) = -\cos x,\ f^{(3)}(x) = \sin x,$$

and after this we repeat these same derivatives: the fourth derivative is $\cos x$, the fifth is $-\sin x$, and so on. Evaluating these are 0 gives

$$f^{(0)}(0) = 1,\ f^{(1)}(0) = 0,\ f^{(2)}(0) = -1,\ f^{(3)}(0) = 0,$$

after which we repeat these values. The Maclauring series of $\cos x$ thus looks like:

$$1 + 0x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = 1 - \frac{x^2}{2!} + \frac{1}{x^4}4! - \frac{1}{6!}x^6 + \cdots.$$

Notice thus the coefficient of any odd power of x in the Maclaurin series is 0, since all of these coefficients come from evaluating $\pm\sin x$ at $x = 0$, and the coefficients of the even powers are all ± 1 divided by an even factorial, since all of these come from evaluating $\pm\cos x$ at $x = 0$.

When writing the Maclaurin series in a nice way it thus makes sense to write it to include only the terms involving an even power of x, since the other terms will all be zero anyway. We thus write the Maclaurin series of $\cos x$ as

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}.$$

Compare this with the infinite sum

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

we wrote out earlier: the point is that the $(-1)^n$ portion of our Maclaurin series expression describes the alternating signs, and the $\frac{x^{2n}}{(2n)!}$ portion describes the fact that we only have even powers of x, and such an even power of x is divided by the factorial of the even number describing that power itself. This Maclaurin series of $\cos x$ is one you should have engrained in your minds and know by heart.

Lecture 16: Taylor Polynomials

Warm-Up. We determine the Taylor series for $f(x) = \sin x$ centered at 0, or in other words its Maclaurin series. The answer will be

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

which you should know by heart. ***TO BE FINISHED***

How to compute sin(1). So now we have the equality

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

which holds for all x since this series (as you can determine) has infinite radius of convergence. Setting $x = 1$ gives the equality

$$\sin 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots .$$

Of course, this infinite sum is not possible to compute directly since we can't sit down and literally add up infinitely many quantities. However, taking finite portions of this sum *will* give approximations to $\sin 1$:

$$\sin 1 \approx 1 - \frac{1}{3!} + \frac{1}{5!}$$

for instance. And this is the point: approximating values of functions using power series is *the* way modern computations of concrete numerical values are actually carried out.

For instance, when I plug $\sin 1$ into my calculator I get:

$$\sin 1 \approx 0.8414709848.$$

How did my calculator come up with this value? The answer is that whomever wrote my phone's calculator software programmed the Maclaurin series for $\sin x$ into the phone, and or at least it programmed as many terms of this series as needed to give approximations good enough to include all the decimal places my phone's screen can actually show! In general, the value of $\sin x$ can be approximated incredibly well by taking enough terms in the Maclaurin series of $\sin x$:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}.$$

The expression on the right is the type of thing my calculator has programmed into its memory, and plugging in various values of x gives very good approximations to $\sin x$.

Taylor polynomials. In general, when we have a function as expressed as power series centered at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

we can approximate f using the first however-many terms we want of this series, and the idea is that taking more and more terms gives better and better approximations. The expression obtained by taking the terms up to the n-th one (i.e. the term involving the n-th power of $x - a$) is called the *n-th degree Taylor polynomial of f centered at a*:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

These polynomials form the basis for all modern numerical computations in TONS of applications.

For instance, recall that the Maclaurin series of e^x is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

The first degree Taylor polynomial centered at 0 gives the approximation

$$e^x \approx 1 + x,$$

the second degree Taylor polynomial gives a better approximation

$$e^x \approx 1 + x + \frac{x^2}{2!},$$

the third degree gives the even better approximation

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!},$$

and so on. Again, next time we'll talk about how to determine just how good these approximations actually are. For now, let me point out that the Wikipedia page for "Taylor series" has a nice animation showing graphically that these polynomials indeed give better and better approximations: the graph for e^x is drawn first, then the graph of $1+x$, then the graph of $1+x+\frac{x^2}{2!}$, and so on, and at each step you can actually see that the Taylor polynomial graphs are getting closer and closer to the actual graph of e^x. Check it out!

Example. We compute the 4-th degree Taylor polynomial of $\frac{1}{3-x}$ centered at 1. We have:

$$f(x) = \frac{1}{3-x},\ f'(x) = \frac{1}{(3-x)^2},\ f''(x) = \frac{2}{(3-x)^3},\ f^{(3)}(x) = \frac{2\cdot 3}{(3-x)^4},\ f^{(4)}(x) = \frac{2\cdot 3\cdot 4}{(3-x)^5}.$$

Thus the required Taylor polynomial is

$$\begin{aligned} &f(1) + f'(1)(x-a) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \\ &= \frac{1}{2} + \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4. \end{aligned}$$

Lecture 17: Taylor Remainders

Warm-Up 1. We find the 3-rd order Taylor polynomial of $f(x) = \sqrt{x}$ centered at 9. We compute:

$$f'(x) = \frac{1}{2\sqrt{x}},\ f''(x) = -\frac{1}{4x^{3/2}},\ f'''(x) = \frac{3}{8x^{5/2}}.$$

Evaluating at the center gives:

$$f'(9) = \frac{1}{6},\ f''(9) = -\frac{1}{4\cdot 27} = -\frac{1}{108},\ f'''(9) = \frac{3}{8\cdot 3^5} = \frac{3}{1944}.$$

Thus the required Taylor polynomial is:

$$\begin{aligned} &f(9) + f'(9)(x-9) + \frac{f''(9)}{2!}(x-9)^2 + \frac{f'''(9)}{3!}(x-9)^3 \\ &= 3 + \frac{1}{6}(x-9) - \frac{1}{2\cdot 108}(x-9)^2 + \frac{3}{6\cdot 1944}(x-9)^3. \end{aligned}$$

Warm-Up 2. ***TO BE FINISHED***

Errors and remainders. Recall the idea that the Taylor polynomials of a function provide approximations to that function:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Let us denote this Taylor polynomial by $T_n(x)$. The *error* arising when using this polynomial to approximate f is given by the difference

$$f(x) - T_n(x),$$

since this tells us precisely how far off the value $T_n(x)$ is from the value $f(x)$. (Actually, since this difference can be positive or negative, we usually care more about the absolute value of this difference $|f(x)-T_n(x)|$.) The difference $f(x)-T_n(x)$ is also called the n-th order Taylor *remainder*.

The key fact which makes Taylor series incredibly worthwhile is that we can actually express this remainder in terms of the function $f(x)$ itself; namely, it is true that we can write this remainder as

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some value of c between x and a. The point is that the error obtained when approximating a function using a Taylor polynomial can be written using the *next higher-order derivative* of f and the next higher-order power of $x - a$. So, if we use the 3-rd degree Taylor polynomial, the error will involve terms of degree (or order) 4, if we use the 6-th degree Taylor polynomial the error will use the 7th-order term, and so on. This is the key observation which allows us to get a handle on how good of an approximation we have.

Example 1. The 3-rd, 4-th, and 5-th degree Taylor polynomials (centered at 0) of e^x are:

$$\begin{aligned}
e^x &\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \\
e^x &\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} \\
e^x &\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.
\end{aligned}$$

Setting $x = 1$ in each gives:

$$\begin{aligned}
e &\approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} \approx 2.67 \\
e &\approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} \approx 2.708 \\
e &\approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.7167.
\end{aligned}$$

The actual value of e is approximately $e \approx 2.71828$, so we indeed see that the approximations given by the Taylor polynomials get better and better as the degree of the polynomial increases.

Let us delve into the 5th degree approximation

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

a little more carefully. This 5-th degree approximation has an error term which can be written as

$$|\underbrace{e^x - (\text{5th degree approximation})}_{\text{error}}| = \left|\frac{f^{(6)}(c)}{6!}x^6\right| = \frac{e^c}{6!}|x|^6$$

for some value of c between 0 (the center) and x. Here we are using the fact that $f^{(6)}(x) = e^x$ for $f(x) = e^x$ in order to get the correct numerator. Since the unknown value of c falls between 0 and x, $e^c < e^x$, we can *bound* this error by:

$$|\text{error}| = \frac{e^c}{6!}|x|^6 \leq \frac{e^x}{6!}|x|^6.$$

To be clear, we are not saying that the error *equals* $\frac{e^x}{6!}|x|^6$, only that it can be no greater than this value. (The actual error involves the unknown value of c.) In particular, for $x = 1$, where we get the approximation

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.7167,$$

the error is bounded by:

$$|\text{error}| \leq \frac{e^1}{6!}1^6 = \frac{e}{720}.$$

That is, the approximation 2.7167 should be within $\frac{e}{720} \approx 0.00378$ of the actual value of e; since $e \approx 2.71828$, we see that the error $2.71828 - 2.7167 = 0.00158$ is indeed less than 0.00378 as expected.

Example 2. Suppose we want to approximate $\sqrt{x}$ near $a = 9$. We can use, say, the 2nd order Taylor polynomial centered at 9, whose terms were computed as part of the Warm-Up:

$$\sqrt{x} \approx 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$$

To be sure, in the Warm-Up we actually computed the 3rd order Taylor polynomial, but truncating that and stopping at the $(x-9)^2$ term gives the 2nd order polynomial above. The error in this approximation can be written as

$$|\text{error}| = \left| \frac{f'''(c)}{3!}(x-9)^3 \right|$$

for some c between x and 9, where $f(x) = \sqrt{x}$. Using the value $f'''(x) = \frac{3}{8x^{5/2}}$ found in the Warm-Up, this gives:

$$|\text{error}| = \frac{3}{3! \cdot 8 \cdot |c|^{5/2}}|x-9|^3.$$

Now, we will in particular approximate $\sqrt{10}$. Since 10 is relatively close to the center 9, we expect the resulting approximation

$$\sqrt{10} \approx 3 + \frac{1}{6}(10-9) - \frac{1}{216}(10-9)^2 = 3 + \frac{1}{6} - \frac{1}{216}$$

to not be so bad. The error is

$$|\text{error}| = \frac{3}{48c^{5/2}}|10-9|^3 = \frac{3}{48c^{5/2}}$$

for some c between 9 and 10. Now, if we want *bound* this error to get rid of the unknown c value, we will need to bound $c^{5/2}$ from *below* since it occurs in the denominator the fraction above. Since $9 < c$, we get $9^{5/2} < c^{5/2}$, so

$$\frac{1}{c^{5/2}} < \frac{1}{9^{5/2}} = \frac{1}{3^5}.$$

Again, we should be clear: we cannot the upper bound of 10 on c since $c^{5/2} < 10^{5/2}$ does NOT give an upper on the reciprocal $\frac{1}{c^{5/2}}$, which is what we need; for this, we need to use the lower bound 9 on c. Thus, we get:

$$|\text{error}| = \frac{3}{48c^{5/2}} \leq \frac{3}{48 \cdot 3^5} \approx 0.000257.$$

That is, the difference between the actual value of $\sqrt{10}$ and the value we get using the approximation

$$\sqrt{10} \approx 3 + \frac{1}{6} - \frac{1}{216}$$

will be less than 0.000257. This is pretty small, so this approximation should be pretty good. Indeed, this approximation roughly gives the value 3.162037, while $\sqrt{10}$ is actually roughly 3.162278, and this difference is roughly

$$3.162278 - 3.162027 = 0.000251,$$

which is indeed less than the estimated error 0.000257 we derived. If nothing else, having an error this small $0.000251 < \frac{1}{10^3}$ implies at least 2 decimal places of accuracy, which is true; in fact, we even get 3 decimal places of accuracy in this case.

Lecture 18: More Series Estimates

Warm-Up. Consider the following 4th degree Taylor approximation of $\cos x$:

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

where the expression on the right is the 4-th degree Taylor polynomial centered at 0. Suppose we use this polynomial to approximate $\cos x$ for values of x in the interval $(-0.5, 0.5)$, or equivalently values of x satisfying $|x| \leq 0.5$. We determine how good of an approximation this is.

Since we approximate using the 4-th degree Taylor polynomial, the error should be given by a term using the 5-th derivative of $f(x) = \cos x$:

$$|\cos x - (\text{4th degree Taylor polynomial})| = \left| \frac{f^{(5)}(c)}{5!} x^5 \right|$$

for some c between 0 and x. Since $f(x) = \cos x$, $f^{(5)}(x) = -\sin c$, so

$$\left| \frac{f^{(5)}(c)}{5!} x^5 \right| = \frac{|-\sin c|}{120} |x|^5.$$

Since $|-\sin c| \leq 1$ and we are considering x satisfying $|x| \leq 0.5$, we get that the error is at most:

$$|\text{error}| \leq \frac{1}{120}(0.5)^5 \approx 0.00026 \leq \frac{1}{10^3} = 0.001.$$

Thus for $|x| \leq 0.5$, $\cos x$ and

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

are within 0.001 of one another, which implies in particular that their values agree to at least 2 decimal places! As a check: plugging $x = 0.25$ into this Taylor approximation gives a value of

$$0.9689176,$$

while the *actual* value of $\cos(0.25)$ is about

$$0.9688912,$$

so indeed these two values agree to at least 2 (in fact more) decimal places. Huzzah, math works!

Example 1. Consider the integral

$$\int_0^1 \sin(x^2)\,dx,$$

which cannot be computed explicitly. However, since

$$\sin y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!},$$

we have

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

Thus

$$\int_0^1 \sin(x^2)\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(4n+3)}.$$

The second degree approximation (going up to $n = 2$) is

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \approx 0.31028.$$

Now, how good is this approximation? This requires estimating the error

$$|(\text{actual integral value}) - (\text{second degree approximation})|.$$

We could do this using the Taylor remainder, but since the series we used to derive this approximation is actually an alternating series, we can also use the alternating series estimate we spoke about briefly when discussin alternating series. Recall that for an alternating series $\sum (-1)^n b_n$, the error obtained in approximating the actual value using the n-th partial sum is at most b_{n+1}:

$$|(\text{actual value}) - (b_0 - b_1 + b_2 + \cdots + (-1)^n b_n)| \le b_{n+1}.$$

In our case, since we used the $n = 2$ partial sum, we get that

$$|(\text{actual integral value}) - (\text{second degree approximation})| \le \frac{1}{75600},$$

where

$$b_n = \frac{1}{(2n+1)!(4n+3)}, \text{ so } b_3 = \frac{1}{75600}.$$

Since

$$\frac{1}{75600} < \frac{1}{10^5},$$

this implies that the actual value of the integral

$$\int_0^1 \sin(x^2)\,dx$$

and the approximation

$$0.31028$$

using the 2-nd order approximation should agree to at least 4 decimal places. Indeed, the actual value of this integral (obtained from my calculator, which itself uses an even better approximation than the one we're using here) is about

$$0.310268,$$

and low-and-behold these two values do agree to 4 decimal places!

Example 2. Let us now approximate the value of the integral

$$\int_0^1 e^{t^2}\, dt,$$

which cannot computed explicitly using standard techniques of integration. We start with:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \rightsquigarrow e^{t^2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}.$$

Integrating gives:

$$\int_0^x e^{t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\, dt = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{n!(2n+1)}\bigg|_0^x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)}.$$

Now, based on this, we can approximate the function $f(x) = \int_0^x e^{t^2}\, dt$ using Taylor polynomials of the form:

$$\int_0^x e^{t^2}\, dt \approx x + \frac{x^3}{3} + \frac{x^5}{2\cdot 5} + \cdots + \frac{x^{2n+1}}{n!(2n+1)}.$$

In particular, say, the 5th order approximation is

$$\int_0^x e^{t^2}\, dt \approx x + \frac{x^3}{3} + \frac{x^5}{10}.$$

Evaluating at $x = 1$ gives

$$\int_0^1 e^{t^2}\, dt \approx 1 + \frac{1}{3} + \frac{1}{10}.$$

The error in this approximation is

$$|\text{error}| = \frac{|f^{(6)}(c)|}{6!}|1-0|^6$$

for some c between 0 and 1. (We need the 6th derivative term since the approximation we are using came from the 5th order Taylor polynomial.) For $f(x) = \int_0^x e^{t^2}\, dt$, we can (tediously) compute that

$$f^{(6)}(x) = (120x + 160x^3 + 32x^5)e^{x^2}, \text{ so } f^{(6)}(c) = (120c + 160c^3 + 32c^5)e^{c^2}.$$

But in our case, c will be between 0 and 1, so we can bound all the c terms by 1 to get:

$$|f^{(6)}(c)| \leq (120 + 160 + 32)e = 312e.$$

Thus our error is

$$|\text{error}| = \frac{|f^{(6)}(c)|}{6!} \leq \frac{312e}{720} \approx 1.178.$$

So, the approximation $1 + \frac{1}{3} + \frac{1}{10} \approx 1.4333$ to the value of $\int_0^1 e^{t^2}\, dt$ is maybe not-so-great in this case, but it is only within 1.178 away from the actual value. (The actual value is something like 1.46265, so our approximation isn't that bad afterall—it's just that the specific way we're looking at for estimating the error overshoots by quite a bit.)

Lecture 19: Complex Numbers

TO BE FINISHED

Lecture 20: Complex Exponentials

TO BE FINISHED

Lecture 21: Second-Order Differential Equations

Warm-Up. ***TO BE FINISHED***

Differential equations. We now shift gears towards studying differential equations. A *differential equation* is an equation which characterizes a function in terms of how it relates to one or more of its derivatives. One example we saw previously was $y' = y$, which characterizes those functions which equal its own derivative. We saw before how we could use series to find the functions which do satisfy this equation, obtaining the fact that y has to look $y = ce^x$ or a constant c. Of course, there are simpler ways of seeing that functions of this form are the only ones which satisfy $y' = y$, but using series to determine such functions—possibly in more complicated cases—will be the final goal of our course.

We will focus on *second-order linear differential equations* with constant coefficients, which are equations of the form

$$ay'' + by' + cy = G(x)$$

where a, b, c are constants and $G(x)$ is some function.

Lecture 22: More on Homogeneous Equations

Lecture 23: Nonhomogeneous Equations

Warm-Up. ***TO BE FINISHED***

Nonhomogeneous solutions. We now consider *nonhomogeneous* second-order linear differential equations with constant coefficients:

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is nonzero. Such equations arise when there is some *external* force or other factor "driving" the behavior of the function y, such as an external motor attached to a spring. (We'll talk about the motion of a spring in more detail later on.)

The key fact, and why we spent time talking about homogeneous equations first, is that all solutions of a nonhomogeneous equation of this type are of the form:

$$y = (\text{homogeneous solution } y_h) + (\text{particular solution } y_p),$$

where the first term is a solution y_h of the corresponding homogeneous equation with $G(x) = 0$ and the second term is a *particular* solution y_p of the given nonhomogeneous equation. (We'll say something about why this works in a bit.) Thus, solving such a differential equation comes down to two steps: first solve the corresponding homogeneous equation, and second find *a* particular solution of the nonhomogeneous equation. Finding a particular solution in the second step will involve coming up with a good guess as to what a solution might look like, and then determining when this guess is in fact valid.

Example 1. We find all solutions of

$$y'' - y' - 2y = x.$$

First, the corresponding homogeneous equation $y'' - y' - 2y = 0$ has general solution given by:

$$y_h = c_1 e^{-x} + c_2 e^{2x},$$

which comes from the fact that the characteristic equation $r^2 - r = 2 = 0$ as roots $r = -1, 2$.

Now we look for *a* (all we need is one!) particular solution of the original nonhomogeneous equation. Given the form of our equation, we should be looking for a function which involves x (or powers of x) in either itself, its derivative, or its second derivative, and polynomials of degree 1 are simple functions with this property. So, we will guess that we can find a particular solution the form

$$y_p = A + Bx$$

with coefficients A and B to-be-determined. (The technique we are using is often called the *method of undetermined coefficients*, since it comes down to computing some coefficients who values are not known initially.) Again, to be clear, at this point this is a merely a *guess* for what a solution might look like, but it is meant to be a reasonable choice for a guess.

The point is that now we can determine what A and B actually need to be in order for this guess to actually work. To say that $y_p = A + Bx$ satisfies the given differential equation is to say that $y_p'' - y_p' - 2y_p = x$, which after computing $y_p'' = 0$ and $y_p' = B$ becomes:

$$\underbrace{0}_{y_p''} - \underbrace{B}_{y_p'} - 2(\underbrace{A + Bx}_{y_p}) = x.$$

This can be rewritten as

$$(-B - 2A) - 2Bx = x,$$

which after comparing like-terms on both sides gives the following requirements:

$$\begin{aligned} -B - 2A &= 0 \\ -2B &= 1. \end{aligned}$$

(The point is that the constants terms on both sides have to match up—where the constant term on the right is 0—and the coefficients of x^1 on both sides have to match up.) The second equation gives $B = -\frac{1}{2}$, and plugging this into the first gives

$$-(-\tfrac{1}{2}) - 2A = 0, \text{ so } A = \tfrac{1}{4}.$$

Thus $y_p = \frac{1}{4} - \frac{1}{2}x$ is our particular solution. The upshot is that by plugging our guess into the differential equation we want it to satisfy, we can work out what the *undetermined coefficients* actually have to be.

Thus, using the particular solution we found, we can conclude that *all* solutions of $y'' - y' - 2y = x$ are given by:

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{4} - \frac{1}{2}x,$$

where we have added the particular solution to the general solution of the homogeneous equation.

Example 2. Now we solve
$$y'' - y' - 2y = x^2.$$
The corresponding homogeneous equation $y'' - y' - 2y = 0$ is the same as in the previous example, where we found the general solution to be
$$y_h = c_1 e^{-x} + c_2 e^{2x}.$$
So again what remains is to find a particular solution of the given nonhomogeneous equation.

With $G(x) = x^2$ on the right, we should expect that our particular solution will be one which gives powers of x when computing derivatives. So, we will take
$$y_p = A + Bx + Cx^2$$
as a guess, where we go up to degree 2 since $G(x) = x^2$ is of degree 2. Now we find the values of A, B, C. In for our guess to actually work, we need $y_p'' - y_p' - 2y_p'$ to equal x^2, which since $y_p'' = 2C$ and $y_p = B + 2Cx$, turns into the following equality:
$$2C - (B + 2Cx) - 2(A + Bx + Cx^2) = x^2.$$
The overall constant term on the left side is $2C - B - 2A$, which has to match up with the constant term of 0 on the right; the coefficient of x on the left is $-2C - 2B$, which has to match up with the coefficient 0 of x n the right (we should interpret the missing x term on the right as $0x$); and the coefficient of x^2 on the left is $-2C$, which has to match up with the coefficient 1 on the right. Thus A, B, C must satisfy the following equations:
$$\begin{aligned} 2C - B - 2A &= 0 \\ -2C - 2B &= 0 \\ -2C &= 1. \end{aligned}$$
The final equation gives $C = -\frac{1}{2}$, then the second gives $1 - 2B = 0$, so $B = \frac{1}{2}$, and then the first gives $-1 - \frac{1}{2} - 2A = 0$, so $A = -\frac{3}{4}$. Thus our particular solution is
$$y_p = -\frac{3}{4} + \frac{1}{2}x - \frac{1}{2}x^2,$$
and hence the general solution to our differential equation is
$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{4} + \frac{1}{2}x - \frac{1}{2}x^2.$$
Why homogeneous plus particular? Finally we explain why, when solving a nonhomogeneous differential equation, we know to expect solutions to look like
$$y = (\text{homogeneous solution } y_h) + (\text{particular solution } y_p).$$

Lecture 24: More on Nonhomogeneous Equations

Warm-Up 1. ***TO BE FINISHED***

Warm-Up 2. Now we solve

$$y'' - y' = x^2.$$

The homogeneous equation $y'' - y' = 0$ in this case has general solution

$$y_h = c_1 + c_2 e^x,$$

which comes from the roots $r = 0, 1$ of the characteristic equation $r^2 - r = 0$. Given the form of $G(x) = x^2$, let us first take the following guess for a particular solution:

$$y_p = A + Bx + Cx^2.$$

This guess will work if it satisfies the following equality:

$$(2C) - (B + 2Cx) = x^2,$$

which comes from computing y_p'', y_p', and plugging them into $y_p'' - y_p'$. But now we run into a problem: there is no way this equality can be achieved, since there is no x^2 term on the left side. We did not run into this problem in the previous example since in that case there was an undifferentiated y term present in the differential equation, which introduced x^2 into the left side. In this case, there is no undifferentiated y in $y'' - y'$, so there is nothing to introduce the required x^2 term.

Thus we must make a different guess—one which will introduce x^2 when computing $y'' - y'$. The simplest way to get this is to start with an x^3 term instead of just x^2 as the highest degree, so we take

$$y_p = Ax + Bx^2 + Cx^3$$

as our guess for a particular solution. Note that we omit a constant term from our guess since it is unnecessary because $y'' - y'$ will not make use of this constant anyway—it will differentiate to zero. In order for this guess to be solution, we need:

$$(2B + 6Cx) - (A + 2Bx + 3Cx^2) = x^2,$$

which results in the following requirements:

$$\begin{aligned} 2B - A &= 0 \\ 6C - 2B &= 0 \\ -3C &= 1. \end{aligned}$$

Solving gives

$$C = -\frac{1}{3} \quad B = \frac{1}{3}C = -\frac{1}{9} \quad A = 2B = -\frac{2}{9},$$

so $y_p = -\frac{2}{9}x - \frac{1}{9}x^2 - \frac{1}{3}x^2$ is a particular solution of our differential equation. The general solution is thus:

$$y = y_h + y_p = c_1 + c_2 e^x - \frac{2}{9}x - \frac{1}{9}x^2 - \frac{1}{3}x^2.$$

Exponential example. Now we consider

$$y'' - y' - 20y = e^{3x}.$$

The solution of the corresponding homogeneous equation, as in the first Warm-Up, is

$$y_h = c_1 e^{-4x} + c_2 e^{5x}.$$

For a particular solution of the nonhomogeneous equation, we make a guess of

$$y_p = Ae^{3x}.$$

This comes from looking for functions which will produce e^{3x} upon taking derivatives or second derivatives, and multiples of e^{3x} itself are such examples.

In order for this guess to work, we need $y_p'' - y_p' - 20y_p$ to equal e^{3x}, which requires:

$$9Ae^{3x} - 3Ae^{3x} - 20Ae^{3x} = e^{3x}.$$

This reduces to $-14Ae^{3x} = e^{3x}$, so

$$-14A = 1, \text{ and thus } A = -\frac{1}{14}.$$

Hence $y_p = -\frac{1}{14}e^{3x}$ is a particular solution of our equation, so the general solution is

$$y = y_h + y_p = c_1 e^{-4x} + c_2 e^{5x} - \frac{1}{14}e^{3x}.$$

Another exponential example. Now suppose we instead had

$$y'' - y' - 20y = e^{5x}.$$

Again the homogeneous solution is $y_h = c_1 e^{-4x} + c_2 e^{5x}$. But now making a guess for a particular solution of $y_p = Ae^{5x}$ will not work: since e^{5x} itself appears as part of the *homogeneous* solution, this guess for y_p will satisfy the homogeneous equation $y_p'' - y_p' - 20y_p = 0$ instead of the nonhomogeneous equation. In other words, there will be no e^{5x} term leftover on the left-hand side to compare to the e^{5x} we want on the right side of the nonhomogeneous equation.

So, we must make a different guess, which in this case is

$$y_p = Axe^{5x}.$$

The idea is that, again we want functions whose derivatives involve e^{5x}, and if multiples of e^{5x} alone are not good enough, then the next best thing is multiples of xe^{5x} since these functions *do* involve e^{5x} as part of their derivatives. We compute:

$$y_p' = Ae^{5x} + 5Axe^{5x} = A(1+5x)e^{5x} \quad \text{and} \quad y_p'' = 5Ae^{5x} + 5A(1+5x)e^{5x} = (10A + 25x)e^{5x}.$$

Thus in order for this guess to work, we need:

$$(10A + 25x)e^{5x} - A(1+5x)e^{5x} - 20Axe^{5x} = e^{5x},$$

which after dividing through by e^{5x} and simplifying, turns into the requirement that

$$9A = 1, \text{ or } A = \frac{1}{9}.$$

Hence $y_p = \frac{1}{9}xe^{5x}$ is a particular solution of our equation, so the general solution is

$$y = y_h + y_p = c_1 e^{-4x} + c_2 e^{5x} + \frac{1}{9}xe^{5x}.$$

Trigonometric example. Finally let us a consider an example with trigonometric $G(x)$, say

$$y'' - y' - 20y = 2\sin 3x.$$

The homogeneous solution is still $y_h = c_1e^{-4x} + c_2e^{5x}$. For a particular solution, we need functions which produce $\sin 3x$ after differentiating once or twice, so let us use

$$y_p = A\cos 3x + B\sin 3x.$$

This gives

$$y_p' = -3A\sin 3x + 3B\cos 3x \quad \text{and} \quad y_p'' = -9A\cos 3x - 9B\sin 3x.$$

Thus y_p satisfies our nonhomogeneous equation when

$$(-9A\cos 3x - 9B\sin 3x) - (-3A\sin 3x + 3B\cos 3x) - 20(A\cos 3x + B\sin 3x) = 2\sin 3x.$$

The overall coefficient of $\cos 3x$ is $-29A - 3B$, which must equal 0 since there is no $\cos 3x$ term on the right side, and the overall coefficient of $\sin 3x$ on the left is $-29B + 3A$, which must match up with the 2 on the right.

Hence we get the equations:

$$\begin{aligned} -29A - 3B &= 0 \\ 3A - 29B &= 1. \end{aligned}$$

Solving these (there is some algebra with not-so-nice numbers here which we'll omit) gives

$$A = \frac{3}{850} \quad \text{and} \quad B = -\frac{29}{850}.$$

Thus $y_p = \frac{3}{350}\cos 3x - \frac{29}{850}\sin 3x$, so the general solution of our nonhomogeneous equation is

$$y = c_1e^{-4x} + c_2e^{5x} + \frac{3}{350}\cos 3x - \frac{29}{850}\sin 3x.$$

Lecture 25: Spring Motion

Warm-Up. ***TO BE FINISHED***

The motion of a spring. ***TO BE FINISHED***

Example. ***TO BE FINISHED***

Simple harmonic motion. ***TO BE FINISHED***

External driving. ***TO BE FINISHED***

Lecture 26: Series Solutions

Warm-Up. ***TO BE FINISHED***

Series solutions. We now come to our final topic for the quarter, that of *series solutions* of differential equations. We actually saw an example of this for the equation $y' = y$ a while back, right before we started talking about complex numbers, where the point was that we are able to determine that the solution y to this equation was of the form $y = ce^x$ by working out what the solution looks like as a series.

This exemplifies the general approach: express the to-be-determined solution as a series $y = \sum c_n x^n$ centered at 0, and determine, based on the fact that this series must satisfy the differential equation in general, what the coefficients c_n actually are and hence what the series must actually look like. In some cases we will be able to determine the series in full explicitly, but at other times the best we will be able to do is work out some specific Taylor polynomial approximation to the sought-after solution.

Example 1. Consider the differential equation

$$y'' - y = 0.$$

We know, using the characteristic equation, that the general is $y = Ae^{-x} + Be^x$ where A, B are arbitrary constants. But, let us now derive this using a series approach. The point here is not that we have no other way of solving this particular equation, but rather that we will work out the series approach in an example to which we already know the answer just to get a sense for how this approach works in general. Next time we will apply this technique to equations we do not yet know how to solve, such as some with non-constant coefficients.

Suppose we express y as a power series centered at 0:

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2},$$

so $y'' - y$, written as a series, looks like:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = (2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots) - (c_0 + c_1 x + x_2 x^2 + \cdots).$$

We want to determine when this will equal 0, since this is the condition needed in order to satisfy $y'' - y = 0$. In order for the expression above to equal 0, the overall constant term has to be zero, the overall coefficient of x has to be zero, the coefficient of x^2 has to be zero, and so on. The overall constant term is $2c_2 - c_0$, the overall coefficient of x is $6c_3 - c_1$, and so on—we get a bunch of individual equations involving the unknown c_n which all have to equal zero.

But we can do this more systematically as follows. Going back to the series expression above for $y'' - y$:

$$y'' - y = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n$$

we want to rewrite it in a way which will make the overall coefficient of x^n clear. The key is that we can rewrite the first sum in terms of x^n instead of x^{n-2} by reindexing:

$$y'' - y = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} - c_n]x^n.$$

In order for this to equal 0, we thus need the coefficient of every x^n to be zero, which gives the following requirement:

$$(n+2)(n+1)c_{n+2} - c_n = 0 \text{ for all } n \geq 0.$$

Solving for c_{n+2} in terms of c_n gives

$$c_{n+2} = \frac{c_n}{(n+2)(n+1)}.$$

Now we get to work! The value of c_0 will be undetermined, but then taking $n = 0$ in the equation above gives

$$c_2 = \frac{c_0}{2}.$$

The value of c_1 will be undetermined, but taking $n = 1$ above gives

$$c_3 = \frac{c_1}{3 \cdot 2}.$$

Taking $n = 2$ gives

$$c_4 = \frac{c_2}{4 \cdot 3},$$

but using the value we've already found for c_2 gives:

$$c_4 = \frac{c_2}{4 \cdot 3} = \frac{c_0}{4 \cdot 3 \cdot 2}.$$

With $n = 3$ we get

$$c_5 = \frac{c_3}{5 \cdot 4},$$

but using the value we already have for c_3 yields

$$c_5 = \frac{c_3}{5 \cdot 4} = \frac{c_1}{5 \cdot 4 \cdot 3 \cdot 2}.$$

And so on: in general, all the even-indexed coefficients can be expressed in terms of c_0, and all the odd-indexed coefficients can be expressed in terms of c_1, and the values we get by following the pattern above are

$$c_{2k} = \frac{c_0}{(2k)!} \quad \text{and} \quad c_{2k+1} = \frac{c_1}{(2k+1)!}.$$

Thus, our solution when written as a power series looks like:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{2k}x^{2k} + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1} \\ &= \sum_{k=0}^{\infty} c_0 \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} c_1 \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

To be clear, in the second line we rewrote our sum by breaking it up into the portion consisting of even powers of x and the portion consisting of odd powers of x:

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots = (c_0 + c_2x^2 + \cdots) + (c_1x + c_3x^3 + \cdots).$$

We did this because the expression we found for c_n itself depends on whether n is even or odd. The expression above hence gives the general solution to $y'' - y = 0$.

But, as stated at the outset, we already know how to solve this equation to get

$$y = Ae^{-x} + Be^x$$

as the general solution, so how exactly does this match up with the solution we found via the series approach? Notice that using $e^x = \sum \frac{x^n}{n!}$ and $e^{-x} = \sum (-1)^n \frac{x^n}{n!}$, we have:

$$e^x + e^{-x} = (1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{3!} + \cdots) + (1 - x + \tfrac{x^2}{2} - \tfrac{x^3}{3!} + \cdots) = 2(1 + \tfrac{x^2}{2!} + \tfrac{x^4}{4!} + \cdots),$$

where all the odd powers cancel out and the even powers double up. Thus

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

In a similar way, computing $e^x - e^{-x}$ gives a sum where all the even powers cancel out and the odd powers double up, which gives

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

So, the series solution we found above can be written as

$$y = c_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + c_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = c_0 \left(\frac{e^x + e^{-x}}{2} \right) + c_1 \left(\frac{e^x - e^{-x}}{2} \right),$$

which after regrouping like terms does match up with $y = Ae^{-x} + Be^x$. The functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh = \frac{e^x - e^{-x}}{2}$$

which are used here are normally called *hyperbolic cosine and sine* respectively, since it turns out they behave in ways similar to the usual cosine and sine functions. So, all we have found here is an alternate way of expressing the solution of $y'' - y = 0$, in terms so-called hyperbolic trig functions. (Again, the point in this example wasn't so much to find the solution using the series approach since we know how to solve $y'' - y = 0$ in a simpler way; rather, the point was just to see an example of the series approach in action.)

Example 2. Now consider the equation

$$y'' - 2y' + y = 0.$$

As in the last example, here we can find the solution using the characteristic equation, which gives

$$y = c_1e^x + c_2xe^x.$$

But, nonetheless, we will see how to solve using the power series approach instead. Suppose we express the to-be-determined solution again as a power series centered at 0:

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

Then, using the expressions for y' and y'' we gave in the previous example, $y'' - 2y' + y = 0$ turns into the following:

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

As before, to determine the overall coefficient of x^n on the left side it will be useful to rewrite the first two terms in terms of x^n instead of x^{n-2} and x^{n-1} respectively by reindexing:

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n - 2\sum_{n=0}(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Thus, the overall coefficient of x^n on the left side, which must be zero in order for this equality to hold, is:

$$(n+2)(n+1)c_{n+2} - 2(n+1)c_{n+1} + c_n = 0.$$

TO BE FINISHED

Lecture 27: More on Series Solutions

Warm-Up. We find the solution of

$$y'' + 16y = 0$$

using the series approach. Suppose

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$y'' + 16y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 16\sum_{n=0}^{\infty} c_n x^x,$$

which we can write as

$$y'' + 16y = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + 16\sum_{n=0}^{\infty} c_n x^n.$$

Thus in order for this to equal 0, we need

$$(n+2)(n+1)c_{n+2} + 16c_n = 0 \text{ for all } n \geq 0.$$

The goal is now to determine what the c_n actually are, to the extent which is possible. When $n = 0$ the recursive equation above becomes:

$$2c_2 + 16c_0 = 0, \text{ which gives } c_2 = \frac{-4^2}{2}c_0.$$

(The reason for writing the coefficient at the end as $-\frac{4^2}{2}$ will become clear in a bit.) When $n = 1$ we get:

$$3 \cdot 2c_3 + 16c_1 = 0, \text{ so } c_3 = \frac{-4^2}{3 \cdot 2} c_1.$$

When $n = 2$ we get

$$4 \cdot 3c_4 + 16c_2 = 0, \text{ so } c_4 = \frac{-4^2}{4 \cdot 3} c_2 = \frac{(-1)^2 4^4}{4 \cdot 3 \cdot 2} c_0,$$

where we replaced c_2 by the value $c_2 = \frac{-4^2}{2}\mathbf{c}_0$ we derived for it before. When $n = 3$ we get

$$5 \cdot 4c_5 + 16c_3 = 0, \text{ so } c_5 = \frac{-4^2}{5 \cdot 4} c_3 = \frac{(-1)^2 4^4}{5 \cdot 4 \cdot 3} c_1.$$

And so on: all the even-indexed coefficients can eventually be written in terms of c_0, and all the odd-indexed ones in terms of c_1. The next few terms look like

$$c_6 = \frac{-4^2}{6 \cdot 5} c_4 = \frac{(-1)^3 4^6}{6!} c_0 \qquad\qquad c_7 = \frac{-4^2}{7 \cdot 6} c_5 = \frac{(-1)^3 4^6}{7!} c_1,$$

and the pattern continues in general: for any n, we have

$$c_{2n} = \frac{(-1)^n 4^{2n}}{(2n)!} c_0 \text{ and } c_{2n+1} = \frac{(-1)^n 4^{2n}}{(2n+1)!} c_1.$$

Thus, if we separate the series defining y into the portion consisting of the even powers of x plus the portion consisting of the odd powers of x, we find that our solution looks like:

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k}}{(2k)!} c_0 x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k}}{(2k+1)!} c_1 x^{2k+1}.$$

Let us rewrite the final sum by increasing the power of 4 by one and in turn dividing by an extra power of 4:

$$\begin{aligned} y &= \sum_{k=0}^{\infty} c_0 (-1)^k \frac{4^{2k} x^{2k}}{(2k)!} + \sum_{k=0} c_1 (-1)^k \frac{4^{2k+1} x^{2k+1}}{4(2k+1)!} \\ &= c_0 \sum_{k=0}^{\infty} (-1)^k \frac{(4x)^{2k}}{(2k)!} + \frac{c_1}{4} \sum_{k=0}^{\infty} (-1)^k \frac{(4x)^{2k+1}}{(2k+1)!}. \end{aligned}$$

The point is that now we can recognize the resulting series as those defining $\cos 4x$ and $\sin 4x$ respectively! Thus, we can write this solution as

$$y = c_0 \cos 4x + \frac{c_1}{4} \sin 4x,$$

which is precisely the solution we expect the equation $y'' + 16y = 0$ to have using roots of the characteristic equation. (Since c_1 is an arbitrary constant, $\frac{c_1}{4}$ is constant as well, so we could just rename this whole thing to be "c_1" and get the form $y = c_0 \cos 4x + c_1 \sin 4x$ of the solution we usually use in the characteristic equation method.) So, we see that we can derive this same general solution using power series instead.

Non-constant coefficient example. Now we consider the equation

$$y'' - 2xy' + 4y = 0.$$

This is NOT an equation we know how to solve yet; the characteristic equation approach is not applicable since this has a non-constant coefficient. Thus, all we have available is the power series approach. So, suppose

$$y = \sum_{n=0}^{\infty} c_n x^n$$

is a solution. Our goal, as usual, is figure out what this series must actually like in order for it to satisfy the differential equation we want.

The equality $y'' - 2xy' + 4y = 0$ becomes

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x \sum_{n=1}^{\infty} nc_n x^{n-1} + 4 \sum_{n=0}^{\infty} c_n x^n = 0.$$

The second term will already involve x^n after we multiply through by the x in front, so this term does not have to be reindexed. The first term does need reindexing, so overall we get:

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - 2 \sum_{n=1}^{\infty} nc_n x^n + 4 \sum_{n=0}^{\infty} c_n x^n = 0.$$

The middle sum thus does not have a constant term since the first term when $n = 1$ involves x to the first power, so the overall constant term on the left side is

$$2c_2 + 4c_0$$

coming from the first and third sums. This must equal 0, so we get $c_2 = -2c_0$. For $n \geq 1$, the overall coefficient of x^n on the left in the series expression above, which must equal zero, involves contributions from each sum, so we get:

$$(n+2)(n+1)c_{n+2} - 2nc_n + 4c_n = 0 \text{ for } n \geq 1.$$

Expressing c_{n+2} in terms of c_n gives:

$$c_{n+2} = \frac{2n-4}{(n+2)(n+1)} c_n.$$

Now we compute some coefficients explicitly. For $n = 1$ we get

$$c_3 = \frac{-2}{3 \cdot 2} c_1.$$

For $n = 2$ we get

$$c_4 = \frac{0}{4 \cdot 3} c_2 = 0.$$

Setting $n = 3$ gives:

$$c_5 = \frac{2}{5 \cdot 4} c_3 = \frac{-2 \cdot 2}{5 \cdot 4 \cdot 3 \cdot 2} c_1,$$

$n = 4$ gives:

$$c_6 = \frac{4}{6 \cdot 7} c_4 = 0$$

and $n = 5$ gives:

$$c_7 = \frac{6}{7 \cdot 6} c_5 = \frac{-2 \cdot 2 \cdot 6}{7!} c_1.$$

In general, all even-indexed terms beyond c_4 will be zero since these can all be eventually expressed in terms of c_4 itself, and the odd-indexed terms are all expressible in terms of c_1 and look like:

$$c_{2n+1} = \frac{-2 \cdot 2 \cdot 6 \cdots (2n-4)}{(2n+1)!} c_1.$$

Thus, the solution y looks like:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\
&= c_0 + c_2 x^2 + \sum_{k=0}^{\infty} \frac{-2 \cdot 2 \cdot 6 \cdots (2n-4)}{(2n+1)!} c_1 x^{2k+1} \\
&= c_0(1-2x^2) + c_1 \sum_{k=0}^{\infty} \frac{-2 \cdot 2 \cdot 6 \cdots (2n-4)}{(2n+1)!} x^{2k+1}.
\end{aligned}$$

In this case it is not easy to simplify the remaining sum in order to get a more easily-recognizable function, but the point is that we still have an explicit series representation of the solution. In particular, note that taking $c_0 = 1, c_1 = 0$ gives $y = 1 - 2x^2$ as one solution, and we can indeed verify that this does satisfy $y'' - 2xy' + 4y = 0$.

Another example. Finally, consider the following differential equation with initial conditions:

$$y'' + y' - x^2 y = 0, \; y(0) = 1, \; y'(0) = 1.$$

We want to determine the 3-rd Taylor approximation to the solution, centered at 0. The point is that, as in the previous example, the solution will not be so straightforward to determine as an explicit series, but we can certainly work out whichever Taylor polynomial we want explicitly.

Suppose we write y as

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

We want to determine the partial sum which goes up to the x^3 term. Right away, the fact that the solution we want is meant to satisfy $y(0) = 1$ and $y'(0) = 1$ gives us two coefficients we want:

$$y(0) = c_0 + c_1 0 + c_2 0 + \cdots = c_0 = 1$$

and

$$y'(0) = c_1 + 2c_2 0 + 3c_3 0 + \cdots = c_1 = 1,$$

so the Taylor polynomial we want so far looks like:

$$y \approx 1 + x + c_2 x^2 + c_3 x^3.$$

To determine c_2, c_3 we use the differential equation y is meant to satisfy. This equation gives the following requirement:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} - x^2 \sum_{n=0}^{\infty} c_n x^n = 0,$$

which when written as a sum looks like:

$$(2c_2 + 6c_3x + 12c_4x^2 + \cdots) + (c_1 + 2c_2x + 3c_3x^2 + \cdots) - (c_0x^2 + c_1x^3 + c_2x^4 + \cdots) = 0.$$

Comparing constant terms on both sides gives:

$$2c_2 + c_1 = 0, \text{ so } c_2 = -\frac{1}{2}c_1 = -\frac{1}{2}.$$

Comparing x^1 terms gives:

$$6c_3 + 2c_2 = 0, \text{ so } c_3 = -\frac{1}{3}c_2 = \frac{1}{6}.$$

This is enough to determine the 3-rd Taylor polynomial of the solution, which is:

$$y \approx 1 + x + c_2x^2 + c_3x^3 = 1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

This is not the actual solution we want, but it is an approximation to the solution.

If we wanted a better approximation, we could next find the value of c_4 in order to get the 4-th order Taylor polynomial. The value of c_4 can be found by considering the x^2 terms in the equality

$$(2c_2 + 6c_3x + 12c_4x^2 + \cdots) + (c_1 + 2c_2x + 3c_3x^2 + \cdots) - (c_0x^2 + c_1x^3 + c_2x^4 + \cdots) = 0.$$

We get:

$$12c_4 + 3c_3 - c_0 = 0, \text{ so } c_4 = \frac{c_0}{12} - \frac{c_3}{4} = \frac{1}{12} - \frac{1}{24} = \frac{1}{24},$$

so the 4-th order Taylor approximation to the solution we want is:

$$y \approx 1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

We could even get a sense of how good this approximation is by working on some bounds on the error, but we won't do that here, and instead will call it a day. Thanks for reading!