1. Determine whether each of the following statements is **TRUE** or **FALSE**. Justify your answer. (This problem has **six** parts.)

   (a) If f is a function of the form $f(x, y) = a + bx + cy + dx^2 + exy + hy^2$ where a, b, c, d, e, h are real numbers and at least one of d, e, h is nonzero, then the level sets of f are either ellipses, hyperbolas, or a pair of crossed lines.

   Answer: FALSE

   If a, b, e, h are all 0, and $c = d = 1$, then
   $f(x, y) = y + x^2$. As in #4 on this exam, the level sets are
   parabolas: $f(x, y) = k \Rightarrow k = y + x^2 \Rightarrow y = k - x^2$, a parabola.

   If a, b, c, e, h are all 0, then
   $f(x, y) = dx^2$. Level sets of this function are either empty, a
   single vertical line, or a pair of vertical lines:
   $k = dx^2 \Rightarrow x = \pm\sqrt{\frac{k}{d}}$
   - empty if $\frac{k}{d} < 0$
   - single line if $k = 0$
   - two lines if $\frac{k}{d} > 0$

   (b) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in $\mathbb{R}^3$. If $\mathbf{a} \times \mathbf{b} - 2\mathbf{c} = \mathbf{0}$, then $\mathbf{a} \cdot \mathbf{c} = 0$.

   Answer: TRUE

   $\vec{a} \times \vec{b} - 2\vec{c} = \vec{0} \quad \Rightarrow \quad (\vec{a} \times \vec{b}) = 2\vec{c}$.

   We know that $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ and to $\vec{b}$,

   So $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

   $\Rightarrow \vec{a} \cdot (2\vec{c}) = 0$

   $\Rightarrow 2\vec{a} \cdot \vec{c} = 0$

   $\Rightarrow \vec{a} \cdot \vec{c} = 0$, as desired.

   Geometrically, $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$, and $\vec{a} \times \vec{b}$ is parallel to $\vec{c}$, so $\vec{a}$ is perpendicular to $\vec{c}$, so $\vec{a} \cdot \vec{c} = 0$.

(c) If $D(\mathbf{f} \circ \mathbf{g})(\mathbf{x})$ is the zero matrix for each $\mathbf{x}$ in the domain of $\mathbf{g}$, then either $\mathbf{f}$ or $\mathbf{g}$ is a constant function.

Answer: FALSE

$$\text{Let } \vec{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ and let } \vec{g}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} \begin{matrix} \text{same functions} \\ \text{different} \\ \text{notation} \end{matrix}$$

$$\text{(or, } f(x,y) = (x,0) \qquad \vec{g}(x,y) = (0,y). \text{)}$$

$$\text{Then } (f \circ g)(x,y) = f(0,y) = (0,0)$$

$$\text{So } D(f \circ g)(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but neither $\vec{f}$ nor $\vec{g}$ is a constant function.

(d) Suppose that $\mathbf{a}$ is a saddle point of a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$. Then there exist linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^2$ such that $Hf(\mathbf{a})\mathbf{v}_1 = c_1\mathbf{v}_1$ and $Hf(\mathbf{a})\mathbf{v}_2 = c_2\mathbf{v}_2$ where c_1 and c_2 are both positive.

Answer: FALSE

By means of contradiction, assume that there exist linearly independent vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^2$ such that $Hf(\vec{a})\vec{v}_1 = c_1\vec{v}_1$ and $Hf(\vec{a})\vec{v}_2 = c_2\vec{v}_2$ where c_1 and c_2 are both positive. Then $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of $Hf(\vec{a})$ with corresponding eigenvalues c_1 and c_2, respectively. Because $c_1 > 0$ and $c_2 > 0$, $Hf(\vec{a})$ is positive definite. By the Second Derivative Test, if $\vec{a}$ is a critical point of f, then f has a local minimum at $\vec{a}$. This contradicts that $\vec{a}$ is a saddle point of f.

(e) If the second order Taylor polynomial of a function $f(x, y, z)$ at the origin is

$$p_2(x, y, z) = 3 - 2x - x^2 - y^2 - z^2,$$

then f has a local maximum at the origin.

Answer: FALSE

$(0, 0, 0)$ is not a critical point of f because

$$\frac{\partial f}{\partial x}(0, 0, 0) = \frac{\partial p_2}{\partial x}(0, 0, 0) = -2 - 2(0) \neq 0.$$

The local extrema of f occur at critical points.

(f) If $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are C^1 functions such that $\nabla f(1, 2, 3) = (1, 2, 3)$ and $\nabla g(1, 2, 3) = (2, 4, 3)$, then $(1, 2, 3)$ is a critical point of f subject to the constraint $g(x, y, z) = g(1, 2, 3)$.

Answer: FALSE

$\nabla f(1, 2, 3) = (1, 2, 3)$ is not parallel to $\nabla g(1, 2, 3) = (2, 4, 3)$.

Therefore, f does not satisfy the Lagrange multiplier condition at $(1, 2, 3)$.

2. Determine whether each of the following statements is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true. Justify your answer. (This problem has **six** parts.)

   (a) For $k > 0$, the surface defined by the spherical equation $\rho = 10 + \sin\phi\cos\theta$ lies between the sphere of radius k centered at the origin and the sphere of radius $2k$ centered at the origin.

   Answer: Sometimes

   Note that, since $-1 \leq \sin\phi\cos\theta \leq 1$, for every point on the surface we have $9 \leq \rho \leq 11$.

   Therefore, the statement is true if, say, $k = 6$, and false if, say, $k = 1$.

   (b) For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ of x and y and differentiable functions of two variables $x = x(s,t)$ and $y = y(s,t)$, we have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$.

   Answer: Sometimes

   True for $f(x,y) \equiv 0$, so $\frac{\partial f}{\partial s} = 0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$.

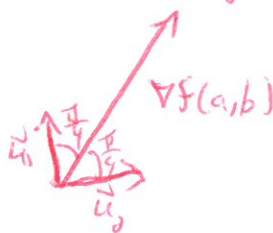   False for $f(x,y) = y$, $y(s,t) = s$, $x(s,t) \equiv 0$

   $$\frac{\partial f}{\partial s} = 1 \neq 0 \cdot 0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$$

(c) For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point (a, b) in $\mathbb{R}^2$, there is an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ of $\mathbb{R}^2$ such that $D_{\mathbf{u}_1} f(a, b) = D_{\mathbf{u}_2} f(a, b)$.

Answer: Always

If $\nabla f(a,b) = \vec{0}$, then any o.n.b. $\{\vec{u}_1, \vec{u}_2\}$ will work.

If $\nabla f(a,b) \neq \vec{0}$, then take $\vec{u}_1, \vec{u}_2$ to be the two unit vectors which make an angle of $45°$ with ∇f.

(d) For a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point $\mathbf{a}$ in $\mathbb{R}^2$ such that $\det Hf(\mathbf{a}) = 0$, f has a saddle point at $\mathbf{a}$.

Answer: SOMETIMES

True example: $f(x,y) = x^4 - y^4$ $\vec{a} = (0,0)$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but $\vec{a}$ is saddle point

False example: $f(x,y) = x^4 + y^4$ $\vec{a} = (0,0)$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but $\vec{a}$ is a local minimum

(e) For a real number k and a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ whose second-order Taylor polynomial at $(0, 1)$ is

$$p_2(x, y) = 5 + 3x^2 + 8x(y - 1) + k(y - 1)^2,$$

the function f has a local minimum at $(0, 1)$.

Answer: SOMETIMES

$$Hf(0,1) = \begin{pmatrix} 6 & 8 \\ 8 & 2k \end{pmatrix}$$

True case: $k = 6$ $\det Hf(0,1) = 8 > 0$
           and $f_{xx}(0,1) = 6 > 0$
           implies local minimum

False case: $k = 0$ $\det Hf(0,1) = -64 < 0$
                       implies saddle point

(f) For a number k, the function $f(x, y, z) = x^2 y^4 + (1 + k^2)z^3$ attains a maximum value over the region described in cylindrical coordinates by $0 \le r \le \sin\theta$.

Answer: NEVER

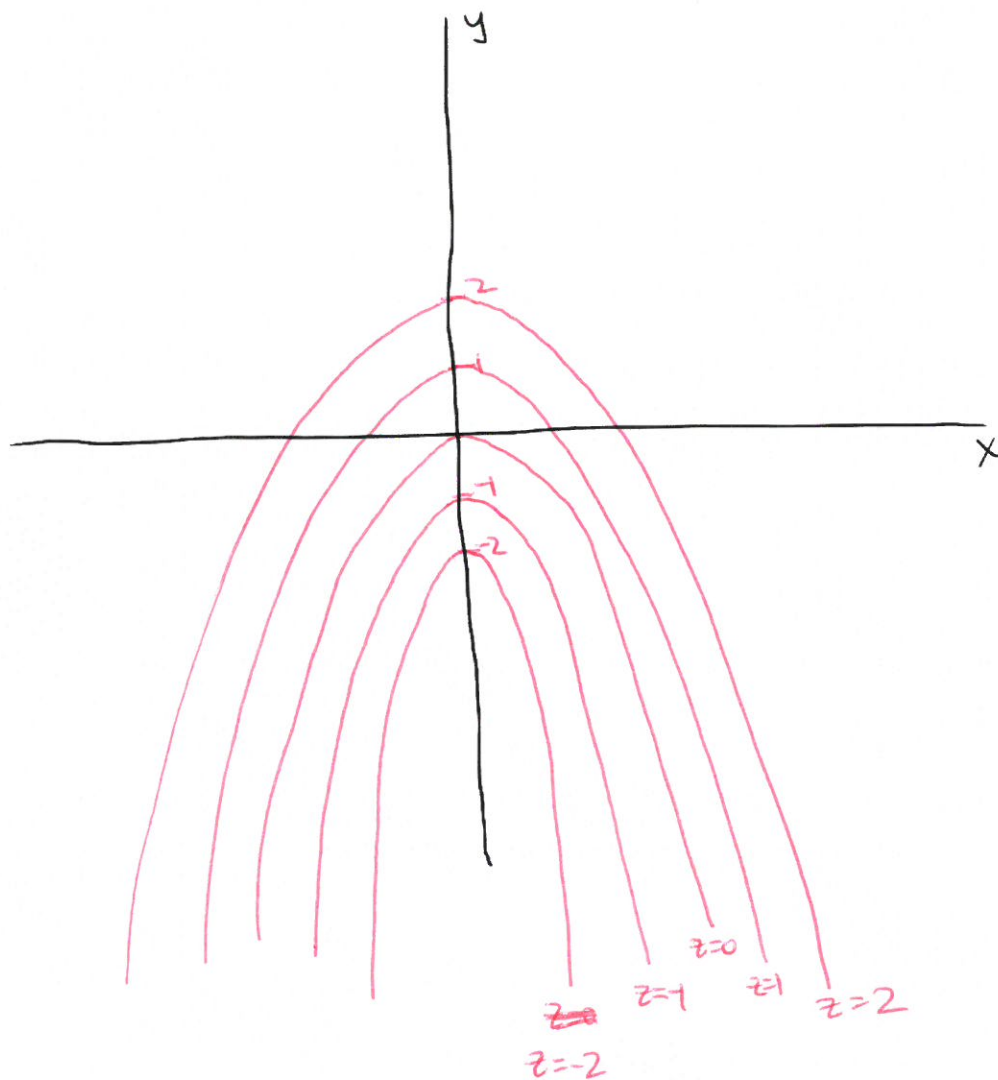region is enclosed by a vertical cylinder

So no bound on z

Since f is unbounded in z-direction

because of $\underbrace{(1+k^2)}_{>0}z^3$ term, no max exists

over this region

3. (This question has **two** parts.) Let f be the function $f(x, y) = x^2 + y$.
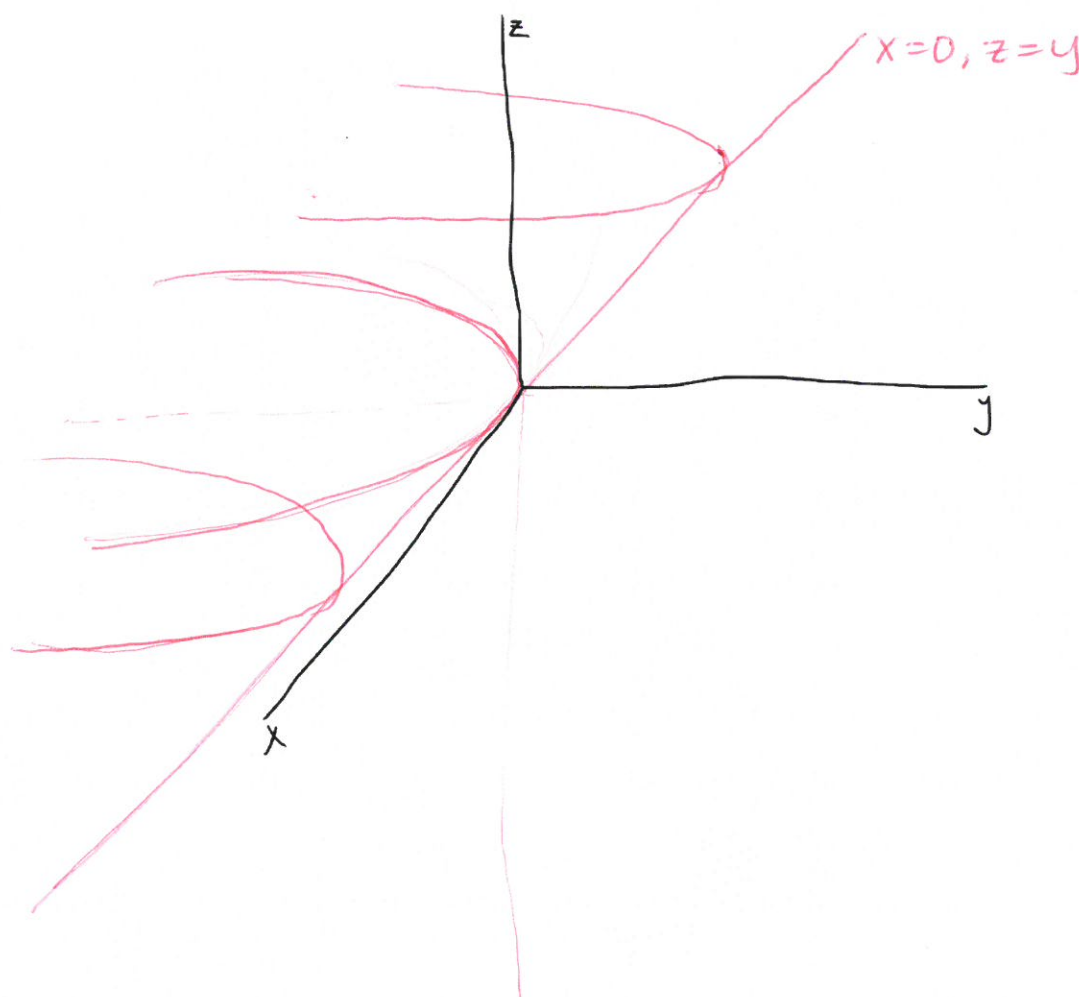
   (a) Sketch at least five level curves of f and label the levels.

Let $z = x^2 + y$

parabolas

(b) Sketch the graph of f. You may use words to help describe your sketch.

z

$x=0, z=y$

y

x

parabolas opening along the negative
y-axis parallel to the xy-plane

4. The tangent plane to a point $(a, b, f(a, b))$ on the graph of $f(x, y) = x^2 + xy + y^2$ is perpendicular to the line given by the parametric equations

$$x = 4 + 8t, \qquad y = 4 + t, \qquad z = -1 - t, \qquad -\infty < t < \infty.$$

Find the values of a and b.

Want the line to be parallel to the normal vector to the tangent plane at $(a, b, f(a,b))$, ie.

$$\langle 8, 1, -1 \rangle = \lambda \langle f_x(a,b), f_y(a,b), -1 \rangle$$
$$= \lambda \langle 2a+b, a+2b, -1 \rangle$$

$$\Rightarrow \lambda = 1 \quad \& \quad \begin{cases} 8 = 2a+b \\ 1 = a+2b \end{cases} \qquad \text{Solve} \sim \boxed{\begin{matrix} b = -2 \\ a = 5 \end{matrix}}$$

5. Let

$$f(u, v) = \frac{u^2 + v^2}{u^2 - v^2}, \quad u(x, y) = e^{-x-y}, \quad \text{and} \quad v(x, y) = e^{xy}$$

Define $h(x, y) = f(u(x, y), v(x, y))$. Find $\partial h/\partial x$. You may leave your answer in terms of $u, v, x,$ and y.

$$\frac{\partial h}{\partial x}(x, y, z) = \frac{\partial f}{\partial u}(u, v)\frac{\partial u}{\partial x}(x, y) + \frac{\partial f}{\partial v}(u, v)\frac{\partial v}{\partial x}(x, y)$$

$$= \frac{2u(u^2 - v^2) - 2u(u^2 + v^2)}{(u^2 - v^2)^2}(-e^{-x-y}) + \frac{2v(u^2 - v^2) + 2v(u^2 + v^2)}{(u^2 - v^2)^2}(ye^{xy})$$

$$= \frac{4uv^2}{(u^2 - v^2)^2}e^{-x-y} + \frac{4u^2v}{(u^2 - v^2)^2}(ye^{xy})$$

$$= \frac{4uv}{(u^2 - v^2)^2}\left(ve^{-x-y} + uye^{xy}\right)$$

6. ConeCo^{TM} manufactures conical waffle cones which are supposed to hold exactly $20\pi/3$ cubic inches of ice-cream. To accomplish this, they have designed their cones to have a base radius of 2 inches and a height of 5 inches. Due to imprecision in the manufacturing process, ConeCo can only guarantee that the radius is between 1.99 and 2.01 inches, and that the height is between 4.95 and 5.05 inches. Use differentials to estimate the maximum error that CupCo can expect in the volume of the cup. (Hint: The volume of a cone with base radius r and height h is $V = \frac{\pi}{3}r^2h$.)

$$dV = \frac{\pi}{3} \cdot 2rh \cdot dr \; + \; \frac{\pi}{3}r^2 \cdot dh$$

$$= \frac{\pi}{3} \cdot 2 \cdot 2 \cdot 5 \cdot 0.01 \; + \; \frac{\pi}{3} \cdot (2)^2 \cdot 0.05$$

$$= \boxed{\frac{2\pi}{15}}$$

7. Find and classify the critical points of the function

$$f(x, y, z) = -x^2 + e^z(z^2 - y^2).$$

① Find the critical points :

Find (x, y, z) : $f_x = -2x = 0$ ↝ $x = 0$

$f_y = -2ye^z = 0$ ↝ $y = 0$

$f_z = 2ze^z + e^z(z^2 - y^2) = 0$ ↝ $ze^z(2 + z) = 0$
$\quad\quad y = 0$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad z = 0, -2$

Crit pts $= \left\{ (0, 0, 0), (0, 0, -2) \right\}$

② Look at Hessian :

$$H_f = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2e^z & -2ye^z \\ 0 & -2ye^z & e^z(-y^2 + 2 + 4z + z^2) \end{pmatrix}$$

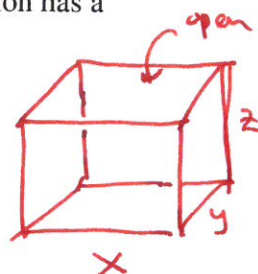$H_f(0,0,0) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ indefinite ⟹ $(0,0,0)$ is a saddle pt

$H_f(0,0,-2) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2e^{-2} & 0 \\ 0 & 0 & -2e^{-2} \end{pmatrix}$ negative definite ⟹ $(0,0,-2)$ is a local max

8. An aquarium in the shape of an open rectangular box without a top is to hold 81 cubic feet of water and is to be built using slate for the rectangular base and glass for the sides. Slate costs \$12 per square foot and glass costs \$2 per square foot. Find the dimensions of the aquarium which minimize the cost. You may assume that the cost function has a global minimum.

$C(x,y,z) = \text{Cost} = 12xy + 4xz + 4yz$

$\text{Constraint} \quad \underbrace{xyz = 81}_{g(x,y,z)}$

$\nabla f = \lambda \nabla g \longrightarrow (12y + 4z, \; 12x + 4z, \; 4x + 4y)$

$$= \lambda (yz, \; xz, \; xy)$$

Solve:

$\left.\begin{array}{l} 12y + 4z = \lambda yz \\ 12x + 4z = \lambda xz \\ 4x + 4y = \lambda xy \end{array}\right\} \quad 12(y-x) = \lambda z (y-x)$

$xyz = 81$

$z \neq 0$ since $xyz = 81$

so $y - x = 0$ ($\lambda = 0$ als. gives $y - x = 0$)

So $\boxed{y = x}$

$4z - 12y = \lambda x (z - 3y)$

$3(z - 3y) = \lambda x (z - 3y)$

$x \neq 0$ since $xyz = 81$

so $z - 3y = 0$

So $\boxed{z = 3y}$

Constraint $\Rightarrow 3x^3 = 81 \Rightarrow x^3 = 27$

$\Rightarrow \boxed{x = 3, \; y = 3, \; z = 9}$

$C(3,3,9) = 324 < C(1,1,81) = 660$ so $(3,3,9)$ not a max