Math 291-1: Final Exam Solutions
Northwestern University, Fall 2016

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If A is a 2×2 matrix such that $A^2 = 0$, then $A = 0$.

(b) If A is a 3×3 matrix whose image is a plane, then A is not invertible.

(c) Any 5 elements of $P_3(\mathbb{R})$ are linearly dependent.

Solution. (a) This is false. For example, take $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

(b) This is true. If the image of A is a plane, then $\operatorname{rank} A = \dim(\operatorname{im} A) = 2$, but an invertible 3×3 matrix must have rank 3.

(c) This is true. $P_3(\mathbb{R})$ is a 4-dimensional vector space, so any number of elements greater than 4 must be linearly dependent. □

2. Let A be an $m \times n$ matrix. Show that the columns of A are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ denote the columns of A. Note that

$$A\mathbf{x} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n.$$

Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$ if and only if the equation

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = 0$$

has only the solution $x_1 = \cdots = x_n = 0$, which is true if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. □

3. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are nonzero linearly independent vectors with the same length and that A is a 2×2 matrix satisfying

$$A\mathbf{v} = \mathbf{w} \text{ and } A\mathbf{w} = \mathbf{v}.$$

Show that A geometrically describes reflection across a line through the origin. Hint: First determine about which line this reflection would have to occur, and then show why A must have the effect of reflecting *any* vector across this line.

Proof. Let L be the line through the origin which bisects the angle between $\mathbf{v}$ and $\mathbf{w}$ and let $R : \mathbb{R}^2 \to \mathbb{R}^2$ denote the linear transformation which reflects a vector $\mathbf{x} \in \mathbb{R}^2$ across L. Since L bisects the angle between $\mathbf{v}$ and $\mathbf{w}$, the angle between $\mathbf{v}$ and L is the same as that between L and $\mathbf{w}$, which implies that reflecting $\mathbf{v}$ gives a vector in the same direction as $\mathbf{w}$ and reflecting $\mathbf{w}$ gives something in the same direction as $\mathbf{v}$. Since $\mathbf{v}$ and $\mathbf{w}$ have the same length, we then have that reflecting $\mathbf{v}$ gives $\mathbf{w}$ and reflecting $\mathbf{w}$ gives $\mathbf{v}$, so $R\mathbf{v} = \mathbf{w}$ and $R\mathbf{w} = \mathbf{v}$.

Since $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ are linearly independent, they must span $\mathbb{R}^2$. So, for a given $\mathbf{x} \in \mathbb{R}^2$, we have

$$\mathbf{x} = c_1\mathbf{v} + c_2\mathbf{w}$$

for some $c_1, c_2 \in \mathbb{R}$, and then

$$A\mathbf{x} = c_1A\mathbf{v} + c_2A\mathbf{w} = c_1\mathbf{w} + c_2\mathbf{v}$$

and

$$Rx = c_1 R\mathbf{v} + c_2 R\mathbf{w} = c_1\mathbf{w} + c_2\mathbf{v}.$$

Thus $A\mathbf{x} = R\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^2$, so $A = R$ as claimed. $\square$

4. Suppose A is an $n \times n$ matrix such that there exists a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ satisfying

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1,\ A\mathbf{v}_2 = \lambda_2\mathbf{v}_2,\ \ldots,\ A\mathbf{v}_n = \lambda_n\mathbf{v}_n$$

for some *nonzero* scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Show that A is invertible.

Proof 1. The given equations can be summarized as

$$A\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix} = \begin{bmatrix}\lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n\end{bmatrix}.$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis for $\mathbb{R}^n$, the matrix $\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}$ is invertible, which gives

$$A = \begin{bmatrix}\lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n\end{bmatrix}\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}^{-1}.$$

We claim that $\lambda_1\mathbf{v}_1, \ldots, \lambda_n\mathbf{v}_n$ are also linearly independent. Indeed, if $c_1, \ldots, c_n \in \mathbb{R}^n$ satisfy

$$c_1(\lambda_1\mathbf{v}_1) + \cdots + c_n(\lambda_n\mathbf{v}_n) = \mathbf{0}$$

, then

$$(c_1\lambda_1)\mathbf{v}_1 + \cdots + (c_n\lambda_n)\mathbf{v}_n = \mathbf{0}$$

and independence of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ gives that $c_i\lambda_i = 0$ for each i; since each λ_i is nonzero, each $c_i = 0$, so $\lambda_1\mathbf{v}_1, \ldots, \lambda_n\mathbf{v}_n$ are linearly independent as claimed. Thus the matrix $\begin{bmatrix}\lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n\end{bmatrix}$ is also invertible, so

$$A = \begin{bmatrix}\lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n\end{bmatrix}\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}^{-1}$$

expresses A as the product of invertible matrices, so it is invertible itself. $\square$

Proof 2. Suppose $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{0}$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$, there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

Then

$$A\mathbf{x} = c_1A\mathbf{v}_1 + \cdots + c_nA\mathbf{v}_n = c_1\lambda_1\mathbf{v}_1 + \cdots + c_n\lambda_n\mathbf{v}_n = \mathbf{0}.$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, we must have each $c_i\lambda_i = 0$, but this implies that each $c_i = 0$ since each λ_i is nonzero. Thus

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

Since the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, A is invertible by the Amazingly Awesome Theorem. $\square$

Proof 3. I'm throwing this in just for fun, but since it uses material on coordinates it was not expected to be the type of solution you should be able to come up with. The idea expressed here, however, is precisely what we'll come back to when discussing eigenvalues, eigenvectors, and the notion of *diagonalization* next quarter.

The matrix of A relative to the basis $\mathscr{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ looks like

$$[A]_{\mathscr{B}} = \begin{bmatrix}[A\mathbf{v}_1]_{\mathscr{B}} & \cdots & [A\mathbf{v}_n]_{\mathscr{B}}\end{bmatrix},$$

where each column is the coordinate vector relative to $\mathscr{B}$ of the result of applying A to a basis vectors in $\mathscr{B}$. Since $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for each i, the coordinate vector of $A\mathbf{v}_i$ has a λ_i in the i-th location and zeroes elsewhere, so

$$[A]_{\mathscr{B}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

is a diagonal matrix with the λ's down the diagonal and zeroes elsewhere. Since each diagonal entry is nonzero, this matrix is invertible (with inverse given by the diagonal matrix with the reciprocals of the λ's down the diagonal), so A is invertible as well. $\square$

5. Let V be a vector space over $\mathbb{K}$. If $v_1, \dots, v_k \in V$ and $w \in \operatorname{span}(v_1, \dots, v_k)$, complete the following proof that

$$\operatorname{span}(v_1, \dots, v_k, w) = \operatorname{span}(v_1, \dots, v_k).$$

(Note: this requires showing that each side is a subset of the other, so if v is in the left side then it is also in the right side, and vice-versa.)

Proof. First let $b \in \operatorname{span}(v_1, \dots, v_k, w)$. Then there exists $a_1, \dots, a_n, a \in \mathbb{K}$ such that

$$b = \underline{a_1 v_1 + \cdots + a_k v_k} + aw.$$

Since $w \in \operatorname{span}(v_1, \dots, v_k)$ we have

$$w = \underline{c_1 v_1 + \cdots + c_k v_k} \text{ for some } c_1, \dots, c_k \in \mathbb{K}.$$

Thus

$$\begin{aligned} b &= \underline{a_1 v_1 + \cdots + a_k v_k} + aw \\ &= a_1 v_1 + \cdots + a_k v_k + \underline{a(c_1 v_1 + \cdots + c_k v_k)} \\ &= \underline{(a_1 + ac_1)} v_1 + \cdots + \underline{(a_k + ac_k)} v_k, \end{aligned}$$

so $b \in \underline{\operatorname{span}(v_1, \dots, v_k)}$.

Conversely suppose $b \in \operatorname{span}(v_1, \dots, v_k)$. Then

$$\underline{b = d_1 v_1 + \cdots + d_k v_k \text{ for some } d_1, \dots, d_k \in \mathbb{K}}.$$

But this can be written as

$$\underline{b = d_1 v_1 + \cdots + d_k v_k + 0w},$$

so $\underline{b \in \operatorname{span}(v_1, \dots, v_k, w)}$. We conclude that $\operatorname{span}(v_1, \dots, v_k, w) = \operatorname{span}(v_1, \dots, v_k)$ as claimed. $\square$

6. Suppose V is a 2-dimensional vector space over $\mathbb{K}$ and that $T : V \to V$ is a linear transformation such that $T^3 = 0$. Show that $T^2 = 0$. Hint: if $v \in V$ is a vector such that $T^2 v \neq 0$, show first that $v, Tv, T^2 v$ are linearly independent.

Proof. Suppose that T^2 is not the zero transformation. Then there exists $v \in V$ such that $T^2 v \neq 0$. Suppose

$$av + bTv + cT^2 v = 0$$

for some $a, b, c \in \mathbb{K}$. Then applying T to both sides gives

$$aTv + bT^2 v = 0$$

where we use the fact that $T^3v = 0$ since $T^3 = 0$ and $T(0) = 0$ since T is linear. Applying T again gives

$$aT^2v = 0$$

for similar reasons. Since $T^2v \neq 0$, this implies that a must be zero. But then the previous equation becomes

$$bT^2v = 0,$$

so $b = 0$ since $T^2v \neq 0$. Finally the original equation becomes $cT^2v = 0$, so $c = 0$ since $T^2v \neq 0$.

Thus v, Tv, T^2v are linearly independent, which is not possible in a 2-dimensional space such as V. Thus T^2 must have been the zero transformation after all. □

7. On the next page is a proof that the dimension of the subspace of $P_3(\mathbb{R})$ consisting of polynomials satisfying $p'(-1) = 0$ is 3 using the rank-nullity theorem. Using this as a guide, do the following. Let $W = \{p(x) \in P_4(\mathbb{R}) \mid p(1) = 0,\ p''(2) = p(1),\text{ and } p'(3) = 0\}$.

(a) Find a linear transformation $T : P_4(\mathbb{R}) \to \mathbb{R}^3$ such that $W = \ker T$.

(b) Find the dimension of W.

Proof for Problem 10. The dimension of the subspace of $P_3(\mathbb{R})$ consisting of polynomials satisfying $p'(-1) = 0$ is 3.

Proof. Define $T : P_3(\mathbb{R}) \to \mathbb{R}$ by

$$T(p(x)) = p'(-1).$$

To be clear, T sends a polynomial to the value of its derivative at -1. This is a linear transformation since it is the composition of the transformation which takes the derivative of a polynomial with the transformation which evaluates a polynomial at -1, both of which are linear. Note that the kernel of T is precisely the subspace in question.

Since $T(x) = 1$, we have $1 \in \operatorname{im} T$ so $\operatorname{im} T$ is at least 1-dimensional. But since $\operatorname{im} T$ is contained in $\mathbb{R}$, we must thus have that $\operatorname{im} T = \mathbb{R}$. Hence $\dim \operatorname{im} T = 1$, so by the rank-nullity theorem we get

$$\dim \ker T = \dim P_3(\mathbb{R}) - \dim \operatorname{im} T = 4 - 1 = 3$$

as claimed. □

Solution to Problem 7. Define $T : P_4(\mathbb{R}) \to \mathbb{R}^3$ by

$$T(p(x)) = \begin{bmatrix} p(1) \\ p''(2) - p(1) \\ p'(3) \end{bmatrix}.$$

Then T is a linear transformation since:

$$T(p(x)+q(x)) = \begin{bmatrix} p(1) + q(1) \\ (p+q)''(2) - (p+q)'(1) \\ (p+q)'(3) \end{bmatrix} = \begin{bmatrix} p(1) \\ p''(2) - p(1) \\ p'(3) \end{bmatrix} + \begin{bmatrix} q(1) \\ q''(2) - q(1) \\ q'(3) \end{bmatrix} = T(p(x))+T(q(x))$$

and

$$T(cp(x)) = \begin{bmatrix} cp(1) \\ (cp)''(2) - (cp)'(1) \\ (cp)'(3) \end{bmatrix} = c \begin{bmatrix} p(1) \\ p''(2) - p(1) \\ p'(3) \end{bmatrix} = cT(p(x)).$$

An element $p(x)$ in the kernel of T satisfies

$$T(p(x)) = \begin{bmatrix} p(1) \\ p''(2) - p(1) \\ p'(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

meaning that $p(1) = 0, p''(2) - p(1) = 0$, and $p'(3) = 0$, which are precisely the requirements needed to belong to W. Thus $\ker T = W$.

(b) By the Rank-Nullity Theorem,

$$\dim(\ker T) = \dim P_4(\mathbb{R}) - \dim(\operatorname{im} T).$$

We know $\dim P_4(\mathbb{R}) = 5$. Since

$$T(1) = \begin{bmatrix} 1 \\ - \\ 0 \end{bmatrix}, \; T(x) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \; T(x^2) = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$$

are three linearly independent vectors in the image of T, this image is at least three dimensional, but then we must have $\operatorname{im} T = \mathbb{R}^3$ since $\mathbb{R}^3$ is 3-dimensional. Thus $\dim(imT) = 3$, so

$$\dim(\ker T) = 5 - 3 = 2,$$

so $\dim W = 2$. $\square$