Math 291-1: Final Exam Solutions
Northwestern University, Fall 2017

1. Determine whether each of the following statements is true or false, and provide justification for your answer.

(a) There is a 3×4 matrix whose columns are linearly independent.

(b) The complex vector space $M_3(\mathbb{C})$ has a 6-dimensional real subspace.

(c) The function $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(p(x)) = (p'(x))^2$ is a linear transformation.

Solution. (a) This is false. The four columns of a 3×4 matrix are vectors in $\mathbb{R}^3$, and any four vectors in a 3-dimensional space must be linearly dependent.

(b) This is true. The set of matrices of the form

$$\begin{bmatrix} a & b & c \\ d & f & g \\ 0 & 0 & 0 \end{bmatrix} \text{ where } a, b, c, d, f, g \in \mathbb{R}$$

is a 6-dimensional real subspace of $M_3(\mathbb{C})$.

(c) This is false. Since $T(2x) = (2)^2 = 4$ and $2T(x) = 2(1)^2 = 2$, we have $T(2x) \neq 2T(x)$ so T does not preserve scalar multiplication. □

2. Consider the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 2 & -2 & 3 & -5 & -1 \\ -3 & 3 & -6 & 12 & 3 \end{array}\right].$$

Find two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^4$ with the property that any solution of the system above can be written as

$$\begin{bmatrix} -3 \\ -2 \\ 2 \\ 1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$. Justify the reason why the vectors you find work.

Proof. Row-reducing the augmented matrix for the corresponding homogeneous system results in the following:

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 2 & -2 & 3 & -5 & 0 \\ -3 & 3 & -6 & 12 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & -3 & 9 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the homogeneous system has solutions given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_4 \\ x_2 \\ 3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix},$$

so $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ span the space of solutions of the homogeneous system. Now, we have:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & -2 & 3 & -5 \\ -3 & 3 & -6 & 12 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3+2+2-1 \\ -6+4+6-5 \\ 9-6-12+12 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix},$$

so $\begin{bmatrix} -3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$ is a particular solution of the given system. Since any solution can be obtained from this one by adding to it a solution of the homogeneous system, we conclude all solutions of the given system are of the form

$$\mathbf{x} = \begin{bmatrix} -3 \\ -2 \\ 2 \\ 1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some $c_1, c_2 \in \mathbb{R}$ where $\mathbf{v}_1, \mathbf{v}_2$ are the vectors above. □

3. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ form a basis of $\mathbb{R}^4$ and that A is a 4×4 matrix with the property that

$$A\mathbf{v}_1 = \mathbf{v}_1,\ A\mathbf{v}_2 = \mathbf{v}_1,\ A\mathbf{v}_3 = \mathbf{v}_2,\ A\mathbf{v}_4 = \mathbf{v}_3.$$

Show that the image of A^4 is the entire span of $\mathbf{v}_1$.

Proof. Let $\mathbf{x} \in \mathbb{R}^n$, and write it as a linear combination of the given basis vectors:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$$

for some $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Then we compute:

$$\begin{aligned} A\mathbf{x} &= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + c_3A\mathbf{v}_3 + c_4A\mathbf{v}_4 = c_1\mathbf{v}_1 + c_2\mathbf{v}_1 + c_3\mathbf{v}_2 + c_4\mathbf{v}_3 \\ A^2\mathbf{x} &= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_1 + c_3A\mathbf{v}_2 + c_4A\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_1 + c_3\mathbf{v}_1 + c_4\mathbf{v}_2 \\ A^3\mathbf{x} &= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_1 + c_3A\mathbf{v}_1 + c_4A\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_1 + c_3\mathbf{v}_1 + c_4\mathbf{v}_1 \\ A^4\mathbf{x} &= (c_1 + c_2 + c_3 + c_4)\mathbf{v}_1. \end{aligned}$$

The vectors $A^4\mathbf{x}$ as $\mathbf{x} \in \mathbb{R}^4$ varies make up all of the image of A^4, so this shows that the image of A^4 is contained in the span of $\mathbf{v}_1$. But also for any $c \in \mathbb{R}$, we have $A(c\mathbf{v}_1) = cA\mathbf{v}_1 = c\mathbf{v}_1$, so any vector in the span of $\mathbf{v}_1$ is contained in the image of A^4. Thus we conclude that $\operatorname{im} A = \operatorname{span}(\mathbf{v}_1)$. □

4. Suppose $A, B \in M_n(\mathbb{R})$. If AB is invertible, show that A and B are each invertible.

Proof. Let $\mathbf{x} \in \mathbb{R}^n$ satisfy $B\mathbf{x} = \mathbf{0}$. Then $A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}$, so $(AB)\mathbf{x} = \mathbf{0}$. Since AB is invertible, this implies $\mathbf{x} = \mathbf{0}$ by the Amazingly Awesome Theorem, so the only solution of $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, and hence B is invertible as well. Then we can write A as

$$A = (AB)B^{-1}.$$

The right side is invertible since it is a product of invertible matrices, so A is invertible too. □

5. Suppose U and W are subspaces of a vector space V over $\mathbb{K}$ which have only the zero vector in common. If $u_1, \dots, u_k \in U$ are linearly independent and $w_1, \dots, w_\ell \in W$ are linearly independent, show that $u_1, \dots, u_k, w_1, \dots, w_\ell$ are linearly independent. (This is not true if U and W have more than the zero vector in common, so you will definitely have to make use of this fact.)

Proof. Suppose $a_1, \dots, a_k, b_1, \dots, b_\ell \in \mathbb{K}$ satisfy

$$a_1u_1 + \cdots + a_ku_k + b_1w_1 + \cdots + b_\ell w_\ell = 0.$$

Then

$$a_1u_1 + \cdots + a_ku_k = -b_1w_1 - \cdots - b_\ell w_\ell.$$

The left side is in U since it is a linear combination of elements of U and U is a subspace of V, while the right side is in W for a similar reason. Thus this is an element which U and W have in common, so it must be the zero vector by our assumptions. Thus

$$a_1u_1 + \cdots + a_ku_k = 0 \quad \text{and} - b_1w_1 - \cdots - b_\ell w_\ell = 0.$$

Since $u_1, \dots, u_k$ are linearly independent, we must then have $a_1 = \cdots = a_k = 0$, and since $w_1, \dots, w_\ell$ are linearly independent, we have $b_1 = \cdots = b_\ell = 0$. We conclude that $u_1, \dots, u_k, w_1, \dots, w_\ell$ are indeed linearly independent. □

6. The *trace* $\operatorname{tr} A$ of a square matrix A is the sum of its diagonal entries:

$$\operatorname{tr}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} := a_{11} + a_{22} + \cdots + a_{nn}.$$

Find a basis for the subspace W of $M_4(\mathbb{R})$ consisting of symmetric matrices of trace zero:

$$W := \{A \in M_4(\mathbb{R}) \mid A^T = A \text{ and } \operatorname{tr} A = 0\}.$$

Don't forget to justify the fact that your claimed basis is actually a basis.

Proof. In order for$A \in M_4(\mathbb{R})$ to satisfy $A^T = A$ it must have the form

$$A = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$$

for some $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$. To satisfy $\operatorname{tr} A = 0$ we also require that

$$a + e + h + j = 0.$$

Thus an element of W concretely looks like:

$$\begin{aligned} A &= \begin{bmatrix} -e-h-j & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} \\ &= b(E_{12} + E_{21}) + c(E_{13} + E_{31}) + d(E_{14} + E_{41}) \end{aligned}$$

$$+ f(E_{23} + E_{32}) + g(E_{24} + E_{42}) + i(E_{34} + E_{43})$$
$$+ e(-E_{11} + E_{22}) + h(-E_{11} + E_{33}) + j(-E_{11} + E_{44})$$

where E_{ij} is the matrix with 1 in row i, column j and zeroes elsewhere. Thus the 9 matrices

$$E_{12}+E_{21}, E_{13}+E_{31}, E_{14}+E_{41}, E_{23}+E_{32}, E_{24}+E_{42}, E_{34}+E_{43}, -E_{11}+E_{22}, -E_{11}+E_{33}, -E_{11}+E_{44}$$

span W. These matrices are linearly independent since if a linear combination of them (with coefficients $b, c, d, f, g, i, e, h, j$ as above) results in the zero matrix, we have

$$\begin{bmatrix} -e-h-j & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which forces $b, c, d, f, g, i, e, h, j$ to all be zero. Hence these 9 matrices form a basis of W. □

7. Let U be the subspace of $P_4(\mathbb{R})$ consisting of all polynomials $p(x) \in P_4(\mathbb{R})$ satisfying both of the conditions

$$p''(2) = p(1) - p(2) \text{ and } p(5) = 0.$$

Determine, with justification, the dimension of U.

Proof. Note first that $p''(2) = p(1) - p(2)$ is equivalent to $p''(2) - p(1) + p(2) = 0$. Define the function $T : P_4(\mathbb{R}) \to \mathbb{R}^2$ by

$$T(p(x)) = \begin{bmatrix} p''(2) - p(1) + p(2) \\ p(5) \end{bmatrix}.$$

Then T is linear, as we can verify directly:

$$\begin{aligned} T(p(x) + q(x)) &= \begin{bmatrix} (p+q)''(2) - (p+q)(1) + (p+q)(2) \\ (p+q)(5) \end{bmatrix} \\ &= \begin{bmatrix} p''(2) + q''(2) - p(1) - q(1) + p(2) + q(2) \\ p(5) + q(5) \end{bmatrix} \\ &= \begin{bmatrix} p''(2) - p(1) + p(2) \\ p(5) \end{bmatrix} + \begin{bmatrix} q''(2) - q(1) + q(2) \\ q(5) \end{bmatrix} \\ &= T(p(x)) + T(q(x)) \end{aligned}$$

and

$$\begin{aligned} T(cp(x)) &= \begin{bmatrix} (cp)''(2) - (cp)(1) + (cp)(2) \\ (cp)(5) \end{bmatrix} \\ &= \begin{bmatrix} cp''(2) - cp(1) + cp(2) \\ cp(5) \end{bmatrix} \\ &= c \begin{bmatrix} p''(2) - p(1) + p(2) \\ p(5) \end{bmatrix} \\ &= cT(p(x)). \end{aligned}$$

The Rank-Nullity Theorem then gives:

$$\dim P_4(\mathbb{R}) = \dim \operatorname{im} T + \dim \ker T.$$

Now, $p(x) \in \ker T$ if and only if

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = T(p(x)) = \begin{bmatrix} p''(2) - p(1) + p(2) \\ p(5) \end{bmatrix},$$

which is true if and only if $p''(2) - p(1) + p(2) = 0$ and $p(5) = 0$. Thus $p(x) \in \ker T$ if and only if $p(x) \in U$, so $U = \ker T$. Since

$$T(1) = \begin{bmatrix} 0 - 1 + 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } T(x) = \begin{bmatrix} 1 - 1 + 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

are both in $\operatorname{im} T$ and are linearly independent, $\operatorname{im} T$ must be at least 2-dimensional. But $\operatorname{im} T$ is a subspace of $\mathbb{R}^2$, so in fact it must equal $\mathbb{R}^2$. Thus:

$$\dim P_4(\mathbb{R}) = \dim \operatorname{im} T + \dim \ker T$$

becomes $5 = 2 + \dim \ker T$, so $\dim \ker T = 3$, and hence ~~$\dim U = 2$~~.

$\dim U = 3$.

□