Math 291-2: Final Exam
Northwestern University, Winter 2017

Name: ______________________________

1. (15 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If a linear transformation preserves the angle between any two vectors, then it is orthogonal.

(b) If $\mathbf{v}$ is an eigenvector of a square matrix A, then $\mathbf{v}$ is also an eigenvector of A^2.

(c) The level curves of $f(x, y) = x^2 - y^2$ are all hyperbolas.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

2. (10 points) Suppose A is an $n \times n$ matrix and that A^T is its transpose.

(a) Show that $A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y}$. Hint: Work out what this becomes when $\mathbf{x} = \mathbf{e}_i$ and $\mathbf{y} = \mathbf{e}_j$.

(b) Show that $(AB)^T = B^T A^T$ for any $n \times n$ matrix B.

3. (10 points) In this problem you can use whichever definition of the determinant you like, but make clear which definition you are using.

(a) Show that a matrix with two identical rows has determinant zero.

(b) Show that the row operation which replaces row $\mathbf{r}_j$ of a matrix by $c\mathbf{r}_i + \mathbf{r}_j$ (where c is a scalar and $\mathbf{r}_i$ is another row) does not change the value of the determinant of that matrix.

4. (10 points) Suppose A is a symmetric 3×3 matrix with eigenvalues $1, 1, -3$ and associated eigenvectors

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \text{ respectively.}$$

(a) Verify that the orthonormal eigenvectors obtained by applying the Gram-Schmidt process to these vectors are:

$$\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \text{ and } \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}.$$

(b) Compute $A^2 \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$.

5. (10 points) Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable and has the property that $Df(\mathbf{x})$ is the same matrix A for every $\mathbf{x}$. Show that f is an affine transformation, i.e. has the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some $\mathbf{b}$. Hint: What is the Jacobian matrix of the function $g(\mathbf{x}) = f(\mathbf{x}) - A\mathbf{x}$ at any $\mathbf{x}$?

6. (10 points) Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are each differentiable and let fg be the function defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$. Complete the following proof of the product rule:

$$D(fg)(\mathbf{x}) = g(\mathbf{x})Df(\mathbf{x}) + f(\mathbf{x})Dg(\mathbf{x}).$$

Proof. Let $h : \mathbb{R}^n \to \_\_$ be the function defined by

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

and $m : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$m(x, y) = \_\_.$$

Then $(fg)(\mathbf{x})$ is the composition ________. By the chain rule we have

$$D(fg)(\mathbf{x}) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

We compute:

$$Dm(x, y) = \begin{bmatrix} y & x \end{bmatrix} \text{ and } Dh(\mathbf{x}) = \begin{bmatrix} \_\_\_\_\_ \\ \_\_\_\_\_ \end{bmatrix},$$

so

$$D(fg)(\mathbf{x}) = \begin{bmatrix} \_\_\_ & \_\_\_ \end{bmatrix} \begin{bmatrix} \_\_\_\_\_ \\ \_\_\_\_\_ \end{bmatrix} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

for any $\mathbf{x} \in \mathbb{R}^n$ as claimed. □

7. (10 points) Let $f(x, y) = xe^{x-y} + 2x^2y$. Suppose that at the point $(1, 1)$ the steepest part of the graph of f has slope M. Find the directions (in terms of explicit vectors) in which the directional derivative of f at $(1, 1)$ is $\frac{M}{\sqrt{2}}$.