Math 291-2: Final Exam
Northwestern University, Winter 2018

Name: ______________________________

1. (15 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If A is a 4×4 matrix whose only eigenvalues are 1 and -3, then A is not diagoanlizable.

(b) There does not exist a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D^2 f(x,y) = \left[\begin{smallmatrix} xy & 2x+y \\ x+2y & y \end{smallmatrix}\right]$.

(c) If $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$, there is no value $f(0,0)$ can have which will make f continuous at $(0,0)$.

Problem	Score
1	
2	
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Total	

2. (10 points) Suppose Q is an $n \times n$ matrix which satisfies

$$Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y} \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Show that Q is orthogonal. You **cannot** take any property of orthogonal matrices for granted here. For instance, if you want to claim that $Q^T Q = I$ implies that Q is orthogonal, you must actually prove this. Here, by "orthogonal matrix" we mean one which has orthonormal columns.

3. (10 points) Let A_n be the $n \times n$ matrix whose entries are all 1's, except for 0's directly below the main diagonal. For instance, A_4 looks like:

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Show that the determinant of A_n is 1 for all n.

4. (10 points) Suppose A is a symmetric $n \times n$ matrix with positive eigenvalues. Show that

$$\mathbf{x} \cdot A\mathbf{x} \geq \lambda_{min} \|\mathbf{x}\|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$, where λ_{min} denotes the smallest eigenvalue of A. Hint: First show this is true for vectors of norm 1 by diagonalizing the quadratic form $\mathbf{x} \cdot A\mathbf{x}$. If you get stuck doing so, take this for granted and use it to do the case of general $\mathbf{x} \in \mathbb{R}^n$.

5. (10 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a function with the property that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0} \text{ for all } \mathbf{x} \in \mathbb{R}^2.$$

Show that f is affine, meaning of the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some 2×2 matrix A and $\mathbf{b} \in \mathbb{R}^2$.

6. (10 points) Suppose $f, g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable and $g(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^2$. Show that the following "quotient rule" holds:

$$D\left(\frac{f}{g}\right)(\mathbf{x}) = \frac{g(\mathbf{x})Df(\mathbf{x}) - f(\mathbf{x})Dg(\mathbf{x})}{g(\mathbf{x})^2} \text{ for all } \mathbf{x} \in \mathbb{R}^2$$

where $\frac{f}{g}$ denotes the function defined by $\mathbf{x} \mapsto \frac{f(\mathbf{x})}{g(\mathbf{x})}$. You **cannot** take the product rule for functions from $\mathbb{R}^n$ to $\mathbb{R}$ for granted, but you can of course prove this first if you need it. Hint: Interpret $\mathbf{x} \to \frac{f(\mathbf{x})}{g(\mathbf{x})}$ as a composition of differentiable functions.

7. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^2y^3 - y\cos(xy)$. If the maximal directional derivative of f in any direction at the point $(\pi, 1)$ is M, determine the explicit directions in which the directional derivative of f at $(\pi, 1)$ is $\frac{-M\sqrt{3}}{2}$.