Math 291-1: Midterm 1 Solutions
Northwestern University, Fall 2017

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample. (A counterexample is a specific example in which the given claim is indeed false.)

(a) If A and B are 2×2 matrices such that $\operatorname{rref}(A) = \operatorname{rref}(B)$ and $A\left[\begin{smallmatrix}\pi\\ e\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$, then $B\left[\begin{smallmatrix}2\pi\\ 2e\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$.

(b) If $\mathbf{w}, \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^2$ and $\mathbf{w}$ is a complex linear combination of $\mathbf{z}_1, \mathbf{z}_2$, then $\mathbf{w}$ is also a real linear combination of $\mathbf{z}_1, \mathbf{z}_2$. (Recall that the distinction between complex and real linear combinations comes in the types of scalars we allow as coefficients.)

Solution. (a) This is true. Since A and B have the same reduced row-echelon form, the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solutions. Thus since $A\left[\begin{smallmatrix}\pi\\ e\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$, it is also true that $B\left[\begin{smallmatrix}\pi\\ e\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$, so

$$B\begin{bmatrix}2\pi\\ 2e\end{bmatrix} = 2B\begin{bmatrix}\pi\\ e\end{bmatrix} = 2\begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

as claimed.

(b) This is false. For an explicit counterexample, take

$$\mathbf{z}_1 = \begin{bmatrix}1\\ 0\end{bmatrix},\ \mathbf{z}_2 = \begin{bmatrix}0\\ 1\end{bmatrix},\ \text{and } \mathbf{w} = \begin{bmatrix}i\\ i\end{bmatrix}.$$

Then $\mathbf{w} = i\mathbf{z}_1 + i\mathbf{z}_2$, so $\mathbf{w}$ is a complex linear combination of $\mathbf{z}_1$ and $\mathbf{z}_2$, but $\mathbf{w}$ is not a real linear combination of $\mathbf{z}_1$ and $\mathbf{z}_2$ since for real a, b,

$$a\begin{bmatrix}1\\ 0\end{bmatrix} + b\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}a\\ b\end{bmatrix}$$

can never equal $\mathbf{w}$. □

2. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^n$ and that $\mathbf{u} \in \mathbb{R}^n$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Show that $\mathbf{u}$ can also be written as a linear combination of

$$\mathbf{v}_1 - \mathbf{v}_3,\ \mathbf{v}_2,\ \mathbf{v}_2 - \mathbf{v}_4,\ \mathbf{v}_3 - \mathbf{v}_4.$$

Proof. Note: This is meant to be similar-in-spirit to the problem on Quiz 2.

Since $\mathbf{u}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, there exist $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4.$$

In order to show that $\mathbf{u}$ is a linear combination of $\mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_3 - \mathbf{v}_4$, we must show there exist $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that

$$\mathbf{u} = a_1(\mathbf{v}_1 - \mathbf{v}_3) + a_2\mathbf{v}_2 + a_3(\mathbf{v}_2 - \mathbf{v}_4) + a_4(\mathbf{v}_3 - \mathbf{v}_4).$$

Define a_1, a_2, a_3, a_4 to be:

$$a_1 = c_1,\ a_2 = c_1 + c_2 + c_3 + c_4,\ a_3 = -c_1 - c_3 - c_4,\ a_4 = c_1 + c_3.$$

Then

$$a_1(\mathbf{v}_1 - \mathbf{v}_3) + a_2\mathbf{v}_2 + a_3(\mathbf{v}_2 - \mathbf{v}_4) + a_4(\mathbf{v}_3 - \mathbf{v}_4)$$

$$
\begin{aligned}
&= c_1(\mathbf{v}_1 - \mathbf{v}_3) + (c_1 + c_2 + c_3 + c_4)\mathbf{v}_2 + (-c_1 - c_3 - c_4)(\mathbf{v}_2 - \mathbf{v}_4) + (c_1 + c_3)(\mathbf{v}_3 - \mathbf{v}_4) \\
&= c_1\mathbf{v}_1 + (c_1 + c_2 + c_3 + c_4 - c_1 - c_3 - c_4)\mathbf{v}_2 + (-c_1 + c_1 + c_3)\mathbf{v}_3 + (-c_1 - c_3 + c_4 + c_1 + c_3)\mathbf{v}_4 \\
&= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \\
&= \mathbf{u}
\end{aligned}
$$

as desired. Hence given $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$, $\mathbf{u}$ is also expressible as a linear combination of $\mathbf{v}_1 - \mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_3 - \mathbf{v}_4$ using the coefficients a_1, a_2, a_3, a_4 defined above.

Note that the values for a_1, a_2, a_3, a_4 come from rewriting

$$a_1(\mathbf{v}_1 - \mathbf{v}_3) + a_2\mathbf{v}_2 + a_3(\mathbf{v}_2 - \mathbf{v}_4) + a_4(\mathbf{v}_3 - \mathbf{v}_4)$$

as

$$a_1\mathbf{v}_1 + (a_2 + a_3)\mathbf{v}_2 + (-a_1 + a_4)\mathbf{v}_3 + (-a_3 - a_4)\mathbf{v}_4$$

and comparing these coefficients to those in $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$; in order to satisfy needed requirements, a_1, a_2, a_3, a_4 should satisfy

$$a_1 = c_1,\ a_2 + a_3 = c_2,\ -a_1 + a_4 = c_3,\ -a_3 - a_4 = c_4,$$

and using this to express a_1, a_2, a_3, a_4 in terms of c_1, c_2, c_3, c_4 gives the values used above. □

3. If $n \geq 2$, show that for any $a \in \mathbb{R}$ and any $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^2$, we have

$$a(\mathbf{v}_1 + \cdots + \mathbf{v}_n) = a\mathbf{v}_1 + \cdots + a\mathbf{v}_n.$$

The only distributive property you can take for granted is that $a(b + c) = ab + ac$ for $a, b, c \in \mathbb{R}$.

Proof. First suppose

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

are two vectors in $\mathbb{R}^2$. Then for any $a \in \mathbb{R}$, we have

$$a(\mathbf{v}_1 + \mathbf{v}_2) = a\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = a\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} a(x_1 + x_2) \\ a(y_1 + y_2) \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + ax_2 \\ ay_1 + ay_2 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ay_1 \end{bmatrix} + \begin{bmatrix} ax_2 + ay_2 \end{bmatrix} = a\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + a\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = a\mathbf{v}_1 + a\mathbf{v}_2.$$

Thus the claimed equality holds for two vectors.

Suppose now that for some $n \geq 2$ the claimed equality holds for any n vectors in $\mathbb{R}^2$, and take any $n + 1$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1} \in \mathbb{R}^2$. Then

$$a(\mathbf{v}_1 + \cdots + \mathbf{v}_{n+1}) = a([\mathbf{v}_1 + \cdots + \mathbf{v}_n] + \mathbf{v}_{n+1}) = a(\mathbf{v}_1 + \cdots + \mathbf{v}_n) + a\mathbf{v}_{n+1}$$

where we use the base case of two vectors. By the induction hypothesis, $a(\mathbf{v}_1 + \cdots + \mathbf{v}_n) = a\mathbf{v}_1 + \cdots + a\mathbf{v}_n$, so

$$a(\mathbf{v}_1 + \cdots + \mathbf{v}_{n+1}) = a(\mathbf{v}_1 + \cdots + \mathbf{v}_n) + a\mathbf{v}_{n+1} = a\mathbf{v}_1 + \cdots + a\mathbf{v}_n + a\mathbf{v}_{n+1}.$$

Hence we conclude by induction that the claimed equality holds for any $n \geq 2$. □

4. Let A be a 4×3 matrix, and let $\mathbf{b}$ and $\mathbf{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about the number of solutions of the system $A\mathbf{x} = \mathbf{c}$? In other words, is it possible for $A\mathbf{x} = \mathbf{c}$ to have no solutions? exactly one solution? infinitely many solutions?

Solution. Note: This was part of Problem 4 on Homework 3, whose solution we reproduce here.

The reduced row-echelon form of $[A \mid \mathbf{b}]$ must look like

$$[A \mid \mathbf{b}] \to \begin{bmatrix} 1 & 0 & 0 & | & ? \\ 0 & 1 & 0 & | & ? \\ 0 & 0 & 1 & | & ? \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

since if there were any free variables the system $A\mathbf{x} = b$ would have infinitely many solutions. Hence the reduced row-echelon form of $[A \mid \mathbf{c}]$ looks like

$$[A \mid \mathbf{c}] \to \begin{bmatrix} 1 & 0 & 0 & | & ? \\ 0 & 1 & 0 & | & ? \\ 0 & 0 & 1 & | & ? \\ 0 & 0 & 0 & | & ? \end{bmatrix},$$

where now we are no longer guaranteed that the lower right entry must be zero—this depends on what $\mathbf{c}$ was to begin with. Hence $A\mathbf{x} = \mathbf{c}$ will have either no solutions or exactly one, depending on what this lower right entry in the reduced form is. $\square$

5. Consider the system of linear equations with augmented matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -1 & 3 \\ -1 & -2 & 0 & -3 & 1 & -2 \\ -2 & -4 & -1 & -8 & 2 & -5 \end{bmatrix}.$$

Show that there exist three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^5$ with the property that any solution $\mathbf{x} \in \mathbb{R}^5$ of this system can be written as

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. (Be sure to explain why the vectors you find are indeed linearly independent!)

Proof. The given augmented matrix describes the linear system with matrix equation

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -1 \\ -1 & -2 & 0 & -3 & 1 \\ -2 & -4 & -1 & -8 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -1 \\ -1 & -2 & 0 & -3 & 1 \\ -2 & -4 & -1 & -8 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$$

is true, the vector

$$\begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

is one particular solution to the given equation. By the result we derived describing the solutions of $A\mathbf{x} = \mathbf{b}$ in terms of those of $A\mathbf{x} = \mathbf{0}$, we know that any solution $\mathbf{x}$ of the given equation can thus be written as

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \mathbf{y}, \text{ where } \mathbf{y} \text{ satisfies } A\mathbf{y} = \mathbf{0}.$$

To solve the equation $A\mathbf{y} = \mathbf{0}$ we row reduce:

$$\begin{bmatrix} 1 & 2 & 1 & 5 & -1 & 0 \\ -1 & -2 & 0 & -3 & 1 & 0 \\ -2 & -4 & -1 & -8 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 5 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Considering x_2, x_4, and x_5 to be free variables, we get that any solution $\mathbf{y}$ is of the form

$$\mathbf{y} = \begin{bmatrix} -2x_2 - 3x_4 + x_5 \\ x_2 \\ -2x_4 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so any solution can be written as a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

These are linearly independent since if $a_1, a_2, a_3 \in \mathbb{R}$ satisfy $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}$, we must have

$$\begin{bmatrix} -2a_1 - 3a_2 + a_3 \\ a_1 \\ -2a_2 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so $a_1 = a_2 = a_3 = 0$. Thus the $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are three linearly independent vectors in $\mathbb{R}^5$ with the property that any solution $\mathbf{x}$ of the original linear system can be written as

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

for some $c_1, c_2, c_3 \in \mathbb{R}$ as required. $\square$