Math 291-2: Midterm 1 Solutions
Northwestern University, Winter 2017

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) A 3×3 matrix with determinant 1 must be orthogonal.

(b) If λ is a real eigenvalue of an orthogonal matrix, then $\lambda = \pm 1$.

Solution. (a) This is false. The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

has determinant 1 but is not orthogonal since not all columns have length 1.

(b) This is true. Suppose λ is a real eigenvalue of an orthogonal matrix Q, and let $\mathbf{v}$ be a corresponding eigenvector. Then $Q\mathbf{v} = \lambda\mathbf{v}$. But since Q is orthogonal we have $\|Q\mathbf{v}\| = \|\mathbf{v}\|$, which means that $|\lambda| = 1$, and hence $\lambda = \pm 1$ as claimed. □

2. Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are orthonormal vectors in $\mathbb{R}^n$. Show that for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n.$$

Proof. Since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are orthogonal, they are linearly independent and hence form a basis for $\mathbb{R}^n$ since $\mathbb{R}^n$ is n-dimensional. Thus for any $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$$

for some $c_1, \ldots, c_n \in \mathbb{R}$. Taking dot products of both sides with $\mathbf{u}_i$ gives

$$\mathbf{x} \cdot \mathbf{u}_i = (c_i\mathbf{u}_i) \cdot \mathbf{u}_i$$

since the dot product of $\mathbf{u}_i$ with all other $\mathbf{u}$'s is zero. Since $\mathbf{u}_i \cdot \mathbf{u}_i = 1$, we get

$$\mathbf{x} \cdot \mathbf{u}_i = c_i(\mathbf{u}_i \cdot \mathbf{u}_i) = c_i,$$

so

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{x} \cdot \mathbf{u}_n)\mathbf{u}_n$$

as claimed. □

3. Find two 3×3 orthogonal matrices Q satisfying

$$Q \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solution. Since Q is meant to be orthogonal, $Q^T = Q^{-1}$ so the given equation can be written as

$$\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = Q^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus the second column of Q^T must be the vector on the left. For the first column of Q^T we can take any vector orthogonal to this given second column, say

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and divide by length to get

$$\begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

Now for the third column we need a vector $\left[\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right]$ which is orthogonal to both of the first two columns, meaning we need

$$-\frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}c = 0 \text{ and } \frac{2}{3}a + \frac{1}{3}b + \frac{2}{3}c = 0,$$

or equivalently

$$-a + c = 0 \text{ and } 2a + b + 2c = 0.$$

Solving this system of equations gives

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

as two possible solutions, and dividing by lengths gives

$$\begin{bmatrix} 1/\sqrt{18} \\ -4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix} \text{ and } \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ -1/\sqrt{18} \end{bmatrix}$$

as two possible third columns of Q^T. Hence taking transposes gives

$$\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 2/3 & 1/3 & 2/3 \\ 1/\sqrt{18} & -4/\sqrt{18} & 1/\sqrt{18} \end{bmatrix} \text{ and } \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 2/3 & 1/3 & 2/3 \\ -1/\sqrt{18} & 4/\sqrt{18} & -1/\sqrt{18} \end{bmatrix}$$

as two possible choices for Q. □

4. Suppose n is odd and that A is an $n \times n$ matrix which is skew-symmetric, meaning $A^T = -A$. Show that A is not invertible. Hint: What is the determinant of A?

Proof. In general $\det(-A) = (-1)^n \det A$, since multiplying A by -1 scales each column by -1, and each such scaling contributes a factor of -1 to $\det(-A)$. Since n is odd, we thus have $\det(-A) = -\det A$. Taking determinants of both sides in $A^T = -A$ thus gives

$$\det A = \det A^T = \det(-A) = -\det A,$$

which implies that $2 \det A = 0$ and hence $\det A = 0$. Thus A is not invertible as claimed. □

5. Let $T : P_6(\mathbb{R}) \to P_6(\mathbb{R})$ be the linear transformation which sends $p(x)$ to $p(-x)$. (To be clear, $p(-x)$ is the polynomial you get by replacing with $-x$ all instances of x in $p(x)$.) Determine the eigenvalues of T and find a basis for each of its eigenspaces.

Proof. For n even we have $T(x^n) = (-x)^n = x^n$, so any such x^n is an eigenvector of T with eigenvalue 1. For n odd, we get $T(x^n) = (-x)^n = -x^n$, so any such x^n is an eivenvector with eigenvalue -1. Thus $1, x^2, x^4, x^6$ are eigenvectors of T with eigenvalue 1 and x, x^3, x^5 are eigenvectors with eigenvalue -1.

These given eigenvectors are all linearly independent and already give the maximum number of linearly independent vectors possible in $P_6(\mathbb{R})$ since $P_6(\mathbb{R})$ is 7-dimensional. Thus there can be no additional eigenvalues since any additional eigenvalues would contribute more linearly independent eigenvectors. In addition, it must be that $1, x^2, x^4, x^6$ span the entire eigenspace corresponding to 1 and that x, x^3, x^5 span the eigenspace corresponding to -1, since if not any additional basis vectors for either of these eigenspaces would give more than 7 linearly independent vectors overall. □