Math 291-3: Midterm 1

Northwestern University, Spring 2018

Name: ______________________________

1. (10 points) Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) If $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable and f has a local maximum at $\mathbf{a}$ among points satisfying $g(\mathbf{x}) = 10$ and $\nabla f(\mathbf{a}) \neq \mathbf{0}$, then there exists $\lambda \in \mathbb{R}$ such that $\nabla g(\mathbf{a}) = \lambda \nabla f(\mathbf{a})$.

(b) The function

$$f(x,y) = \begin{cases} \frac{1}{xy} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is integrable over any square.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Consider rectangles of a fixed area $A > 0$ whose sides have length at most $100\sqrt{A}$. Show that among all such rectangles there is one of minimal perimeter and determine this minimal perimeter.

3. (10 points) Let D be the region $\mathbb{R}^2$ enclosed by the circle with equation $(x-4)^2+(y-\frac{1}{2})^2 = \frac{1}{16}$. Show that

$$\frac{3\pi}{8} \leq \iint_D \left(xy + \frac{8}{x} + \frac{1}{y} \right) \, dA.$$

You may assume that the global minimum of the integrand does not occur on the boundary of D.

4. (10 points) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. Rewrite the following iterated integral with respect to the order $dy\,dx\,dz$ instead.

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

5. (10 points) Let D be the region in $\mathbb{R}^2$ enclosed by the circle with equation $x^2 + y^2 = 4x$. Show that

$$\iint_D (y^{101} + \sqrt{x^2 + y^2})\, dA = \int_{-\pi/2}^{\pi/2} \frac{64}{3} \cos^3\theta \, d\theta.$$

You do NOT have to compute the integral on the right.