Math 291-1: Midterm 2 Solutions
Northwestern University, Fall 2017

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) There is no 2×2 matrix A such that $A^2 \neq I$ but $A^4 = I$.

(b) There is no vector space over $\mathbb{C}$ which has dimension 5 over $\mathbb{R}$.

Solution. (a) This is false. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which describes rotation by $\frac{\pi}{2}$. Then $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is not the identity but $A^4 = (A^2)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is.

(b) This is true: if V is n-dimensional over $\mathbb{C}$, then it is $2n$-dimensional over $\mathbb{R}$ so this dimension over $\mathbb{R}$ can never be odd. (This uses the fact that if $v_1, \ldots, v_n$ forms a basis for V over $\mathbb{C}$, then $v_1, iv_1, \ldots, v_n, iv_n$ forms a basis for V over $\mathbb{R}$ as shown on a homework problem. □

2. Suppose A is an $n \times n$ matrix and that $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ are linearly independent vectors such that

$$A\mathbf{v}_1 = \mathbf{v}_2,\ A\mathbf{v}_2 = \mathbf{v}_3,\ \ldots,\ A\mathbf{v}_{n-1} = \mathbf{v}_n, \text{ and } A\mathbf{v}_n = \mathbf{v}_1.$$

To be clear, A has the effect of sending each of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to the next vector in the list, except that $\mathbf{v}_n$ is sent to $\mathbf{v}_1$. Show that $A^n = I$.

Proof. First we claim that $A^n\mathbf{v}_i = \mathbf{v}_i$ for each i. Indeed, note that multiplication by A increases the index of $\mathbf{v}_i$ for $i = 1, \ldots, n-1$ by 1, so multiplication by A^2 shifts the index of $\mathbf{v}_i$ for $i = 1, \ldots, n-2$ by 2, and so on. This implies that

$$A^{n-i}\mathbf{v}_i = \mathbf{v}_n$$

for $i = 1, \ldots, n$. Thus for each $i = 1, \ldots, n$:

$$A^n\mathbf{v}_i = A^i A^{n-1}\mathbf{v}_i = A^i\mathbf{v}_n = A^{i-1}\mathbf{v}_1 = \mathbf{v}_{1+i-1} = \mathbf{v}_i$$

as claimed.

Now, since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent vectors in the n-dimensional space $\mathbb{R}^n$, they automatically space $\mathbb{R}^n$. Thus if $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

for some $c_1, \ldots, c_n \in \mathbb{R}$. Hence

$$A^n\mathbf{x} = c_1A^n\mathbf{v}_1 + \cdots + c_nA^n\mathbf{v}_n = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{x},$$

so A^n sends every $\mathbf{x} in \mathbb{R}^n$ to itself and thus $A^n = I$ as desired. □

3. Suppose A and B are $n \times n$ matrices such that $AB = I_n$. Show that A and B are each invertible. (You cannot use the fact that if $AB = I_n$ for square matrices, then $BA = I_n$ automatically since that fact relies on the claim given here. You also cannot use the fact that if AB is invertible, then A and B are each invertible, since this also relies on the claim given here.) Hint: Show that B is invertible first, using some aspect of the Amazingly Awesome Theorem.

Proof. Suppose $\mathbf{x} \in \mathbb{R}^n$ satisfies $B\mathbf{x} = \mathbf{0}$. Then multiplying both sides by A on the left gives

$$AB\mathbf{x} = A\mathbf{0} = \mathbf{0}.$$

Since $AB = I$ this implies that $\mathbf{x} = \mathbf{0}$, so the only solution to $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Thus B is invertible by the Amazingly Awesome Theorem. Then multiplying both sides of $AB = I$ by B^{-1} on the right gives

$$ABB^{-1} = IB^{-1}, \text{ so } A = B^{-1}.$$

Hence A is invertible as well since it is the inverse of an invertible matrix. $\square$

4. Suppose V is a vector space over $\mathbb{K}$ and that U is a (linear) subspace of V. Suppose $b \in V$ is not in U, and define $b + U$ to be the set of all vectors in V obtained by adding V to elements of U:

$$b + U = \{b + u \in V \mid u \in U\}.$$

Let $w_1, \ldots, w_k \in b + U$ and $c_1, \ldots, c_k \in \mathbb{K}$. Show that $c_1w_1 + \cdots + c_kw_k \in b + U$ if and only if $c_1 + \cdots + c_k = 1$. (You cannot take it for granted that $b + U$ is an affine subspace of V, since this fact is a consequence of this problem.)

Be careful: the forward direction, namely that if $c_1w_1 + \cdots + c_kw_k \in b + U$ then $c_1 + \cdots + c_k = 1$, is not as obvious as it seems and requires some real thought.

Proof. Suppose $c_1w_1 + \cdots + c_kw_k \in b + U$. Write each w_i as

$$w_i = b + u_i \text{ for some } u_i \in U.$$

Then

$$c_1w_1 + \cdots + c_kw_k = c_1(b + u_1) + \cdots + c_k(b + u_k) = (c_1 + \cdots + c_k)b + (c_1u_1 + \cdots + c_ku_k).$$

Since this is assumed to be in $b + U$, we have

$$(c_1 + \cdots + c_k)b + (c_1u_1 + \cdots + c_ku_k) = b + u$$

for some $u \in U$. (The subtlety here is that at this point u is *not* necessarily the same as $c_1u_1 + \cdots + c_ku_k$, so we can't say right away that $(c_1 + \cdots + c_k)b$ is the same as b and hence that $c_1 + \cdots + c_k$ equals 1. We need more work to get to this point.) Rewriting the expression above gives

$$(c_1 + \cdots + c_k - 1)b = u - (c_1u_1 + \cdots + c_ku_k).$$

Since U is a subspace of V, $c_1u_1 + \cdots + c_ku_k \in U$ since U is closed under linear combinations and hence $u - (c_1u_1 + \cdots + c_ku_k) \in U$ as well. This then implies that $(c_1 + \cdots + c_k - 1)b \in U$, so if $c_1 + \cdots + c_k - 1 \neq 0$ this would give that $b \in U$ after multiplying through by $\frac{1}{c_1 + \cdots + c_k - 1}$. Since $b \notin U$, it must thus be the case that $c_1 + \cdots + c_k - 1 = 0$, so $c_1 + \cdots + c_k = 1$ as claimed.

Conversely suppose that $c_1 + \cdots + c_k = 1$. Again write each w_i as

$$w_i = b + u_i \text{ for some } u_i \in U,$$

so that

$$c_1w_1 + \cdots + c_kw_k = (c_1 + \cdots + c_k)b + (c_1u_1 + \cdots + c_ku_k).$$

Since $c_1 + \cdots + c_k = 1$, this becomes $b + (c_1u_1 + \cdots + c_ku_k)$, and since U is a subspace of V we have $c_1u_1 + \cdots + c_ku_k$ since subspaces are closed under linear combinations. Thus

$$c_1w_1 + \cdots + c_kw_k = b + (c_1u_1 + \cdots + c_ku_k)$$

is in $b + U$ as required. $\square$

5. Let U be the subset of $M_2(\mathbb{C})$ consisting of all 2×2 complex matrices which equal their own transpose:

$$U := \{A \in M_2(\mathbb{C}) \mid A^T = A\}.$$

Show that U is a subspace of $M_2(\mathbb{C})$ over $\mathbb{R}$, and find a basis for U over $\mathbb{R}$. You can take it for granted that $(A+B)^T = A^T + B^T$ and $(cA)^T = cA^T$, where c is a scalar. You do NOT have to justify the fact that your claimed basis is actually a basis.

Solution. First, the tranpose of the zero matrix is itself, so $0 \in U$. If $A, B \in U$, then $A^T = A$ and $B^T = B$, so

$$(A+B)^T = A^T + B^T = A + B.$$

Thus $A + B \in U$, so U is closed under addition. Finally, if $c \in \mathbb{R}$ and $A \in U$, we have

$$(cA)^T = cA^T = cA$$

since $A^T = A$. Thus $cA \in U$, so U is closed under scalar multiplication. We conclude that U is a subspace of $M_2(\mathbb{C})$ over $\mathbb{R}$ as claimed.

In order for $\left[\begin{smallmatrix} a+ib & c+id \\ p+iq & s+it \end{smallmatrix}\right]$ to belong to U requires that

$$\begin{bmatrix} a+ib & p+iq \\ c+id & s+it \end{bmatrix} = \begin{bmatrix} a+ib & c+id \\ p+iq & s+it \end{bmatrix},$$

so $p + iq = c + id$. Thus an element of U can be written as

$$\begin{aligned} \begin{bmatrix} a+ib & c+id \\ c+id & s+it \end{bmatrix} &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &\quad + d \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + s \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 * i & \end{bmatrix}. \end{aligned}$$

Hence the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$$

span U and since they are linearly independent over $\mathbb{R}$, they form a basis for U. $\square$