Math 291-2: Midterm 2 Solutions
Northwestern University, Winter 2017

1. Determine whether each of the following statements is true or false. If it is true, explain why; if it is false, give a counterexample.

(a) There is a 2×2 symmetric matrix A such that $A\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = 2\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] = 3\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$.

(b) There is a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Df(\mathbf{x}) = \left[\begin{smallmatrix}1 & 2\\-1 & 1\end{smallmatrix}\right]$ for every $\mathbf{x} \in \mathbb{R}^2$.

Solution. (a) This is false. Note that this given equalities say that $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$ would be eigenvectors of A corresponding to 2 and 3 respectively. If there were such a symmetric matrix, $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ would have to be orthogonal, which they are not, since eigenvectors corresponding to distinct eigenvalues of a symmetric matrix must always be orthogonal.

(b) This is true: the function $f(x, y) = (x + 2y, -x + y)$ works. This function is differentiable since the component functions $x + 2y$ and $-x + y$, being polynomials, are differentiable. Also, the partial derivatives of the first components are 1 and 2, and those of the second component are -1 and 1, so the Jacobian matrix of this function at any $\mathbf{x}$ is indeed $\left[\begin{smallmatrix}1 & 2\\-1 & 1\end{smallmatrix}\right]$. □

2. Determine the values of k for which the following matrix is diagonalizable. The eigenvalues are $k, 1$, and -3.

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 3 & k & 3 \\ 1 & 0 & -2 \end{bmatrix}$$

Solution. (Note that this was essentially the first problem from Discussion 3.) This matrix has either two or three distinct eigenvalues depending on what k is. If $k \neq 1, -3$, there are three distinct eigenvalues and so in this case A is for sure diagonalizable: with three distinct eigenvalues each eigenspace is 1-dimensional and finding a basis vector for each gives 3 linearly independent eigenvectors overall.

If $k = 1$, then there are only two eigenvalues: 1 with algebraic multiplicity 2 and -3 with algebraic multiplicity 1. We will get one basis eigenvector corresponding to -3, so what determines whether or not A is diagonalizable is how many basis eigenvectors we get for the eigenvalue 1. We have (keeping in mind that $k = 1$):

$$A - I = \begin{bmatrix} -1 & 0 & 3 \\ 3 & 0 & 3 \\ 1 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 12 \\ 0 & 0 & 0 \end{bmatrix},$$

so E_1 is 1-dimensional. Hence we only get one basis eigenvector for $\lambda = 1$, and together with the basis eigenvector for -3 we only get two overall, so A is not diagonalizable.

If $k = -3$, then again there are two eigenvalues, but now 1 has algebraic multiplicity 1 and -3 has algebraic multiplicity 2. We will get one basis eigenvector corresponding to 1, and since (keeping in mind that $k = -3$)

$$A + 3I = \begin{bmatrix} 3 & 0 & 3 \\ 3 & 0 & 3 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has a 2-dimensional kernel, E_{-3} is two dimensional so we get two basis eigenvectors. These together with the basis eigenvector for 1 gives three in total, so A is diagonalizable.

To summarize, A is diagonalizable for all $k \neq 1$. Note however that the reasons differ for $k \neq -3$ and $k = -3$: in the former case there are three distinct eigenvalues, while in the latter there are only two but the geometric multiplicity of each eigenvalue agrees with its algebraic multiplicity. □

3. For fixed k, determine whether the surface in $\mathbb{R}^3$ with equation

$$3x^2 - y^2 + 3z^2 + 2xz = k$$

is an ellipsoid, a double cone, a one-sheeted hyperboloid, or a two-sheeted hyperboloid. The answer will depend on k.

Solution. (Note that was essentially the second problem from Discussion 4.) First we rewrite the left-hand side in terms of new coordinates. This left-hand side is a quadratic form with symmetric matrix

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

After finding eigenvalues and eigenvectors, this can be orthogonally diagonalized as:

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{3} & -1/\sqrt{2} \\ 1 & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{3} & -1/\sqrt{2} \\ 1 & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}^T.$$

After picking coordinates c_1, c_2, c_3 with respect to the orthonormal eigenvectors making up the columns of the first matrix above, the given equation becomes:

$$-c_1^2 + 4c_2^2 + 2c_3^2 = k.$$

For $k > 0$ the surface is a hyperboloid of one sheet centered along the c_1-axis with level curves the ellipses in the c_2, c_3-plane given by

$$4c_2^2 + 2c_3^2 = k + c_1^2.$$

For $k = 0$ the level surface is the double cone centered along the c_1-axis given by

$$4c_2^2 + 2c_3^2 = c_1^2.$$

For $k < 0$, the level surface is the hyperboloid of two sheets centered along the c_1-axis with level curves the ellipses in the c_2, c_3-plane given by

$$4c_2^2 + 2c_3^2 = k + c_1^2.$$

This hyperboloid of two sheets intersects the c_1-axis at $c_1 = \pm\sqrt{k}$ (recall that $k < 0$ here), and no portion of this hyperboloid lies between these values of c_1. $\square$

4. Recall that $U \subseteq \mathbb{R}^2$ is open if for any $\mathbf{p} \in U$, there exists $r > 0$ such that $B_r(\mathbf{p}) \subseteq U$. Show, using this definition, that the region

$$\{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1\}$$

is open. (This is the square with vertices at $(-1,-1), (-1,1), (1,-1)$, and $(1,1)$, only with the corners and sides of the square excluded.)

Proof. Denote the given square by U and let $(p,q) \in U$. Then $-1 < p < 1$ and $-1 < q < 1$, so $|p| < 1$ and $|q| < 1$. Thus $1 - |p|$ and $1 - |q|$ are both positive, so their minimum

$$r = \min\{1 - |p|, 1 - |q|\}$$

is positive as well. We claim that for this radius, $B_r(p,q) \subseteq U$, which will show that U is open.

Indeed, let $(x,y) \in B_r(p,q)$, Then the distance between x and p is no more than r, and the distance between y and q is no more than r. Thus

$$|x| = |x - p + p| \le |x - p| + |p| < r + |p| \le (1 - |p|) + |p| = 1$$

and

$$|y| = |y - q + q| \le |y - q| + |q| < r + |q| \le (1 - |q|) + |q| = 1.$$

Hence $-1 < x < 1$ and $-1 < y < 1$, so $(x,y) \in U$ and thus $B_r(p,q) \subseteq U$ as claimed. □

5. Determine whether or not the following function is differentiable at $(0,0)$.

$$f(x,y) = \begin{cases} y - \frac{3x^3 - 2y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution. First we compute the partial derivatives of f at $(0,0)$. We have

$$f(x,0) = -\frac{3x^3}{x^2} = -3x \text{ for all } x \text{ and } f(0,y) = y - \frac{-2y^4}{y^2} = y + 2y^2 \text{ for all } y.$$

Differentiating these single-variable functions and evaluating at 0 gives $f_x(0,0) = -3$ and $f_y(0,0) = 1$, so $Df(0,0) = \begin{bmatrix} -3 & 1 \end{bmatrix}$.

Now we compute:

$$\begin{aligned} f(h,k) - f(0,0) - Df(0,0)\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] &= k - \frac{3h^3 - 2k^4}{h^2+k^2} + 3h - k \\ &= \frac{k(h^2+k^2) - 3h^3 + 2k^4 + 3h(h^2+k^2) - k(h^2+k^2)}{h^2+k^2} \\ &= \frac{2k^4 + 3hk^2}{h^2+k^2}. \end{aligned}$$

Thus

$$\frac{f(h,k) - f(0,0) - Df(0,0)\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]}{\|(h,k)\|} = \frac{2k^4 + 3hk^2}{(h^2+k^2)^{3/2}}.$$

Converting to polar coordinates using $h = r\cos\theta, k = r\sin\theta$ gives

$$\frac{2r^4\sin^4\theta + 3r^3\cos\theta\sin^2\theta}{r^3} = 2r\sin^4\theta + 3\cos\theta\sin^2\theta.$$

Taking the limit as $r \to 0$ along $\theta = 0$ gives a different value than that along $\theta = \pi/4$, so

$$\lim_{r\to 0} \frac{2r^4\sin^4\theta + 3r^3\cos\theta\sin^2\theta}{r^3}$$

does not exist. Hence

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - Df(0,0)\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]}{\|(h,k)\|}$$

does not exist either, so f is not differentiable at $(0,0)$. □