Math 300: Final Exam
Northwestern University, Spring 2017

Name: ______________________________

1. (10 points) Give an example of each of the following with brief justification.
(a) A true implication $P \Rightarrow Q$ whose converse $Q \Rightarrow P$ is false.
(b) A function $f : \mathbb{R} \to \mathbb{R}$ which is injective but not surjective.
(c) A countable subset of the power set of $\mathbb{R}$.

Problem	Score
1	
2	
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Total	

2. (a) (5 points) Show that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. You may take for granted the fact that for any $x \in \mathbb{R}$, there exists $M \in \mathbb{N}$ such that $x < M$.

(b) (5 points) Show that if $x \leq y + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $x \leq y$.

3. (10 points) Let A and B be sets. Show that

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

(This is known as the *symmetric difference* of A and B, and consists of all elements which belong to either A or B, but not both.)

4. (10 points) Suppose $f : A \to B$ is a function and that S is a subset of B.
(a) Show that $f(f^{-1}(S)) \subseteq S$.
(b) Show that if f is surjective, then $f(f^{-1}(S)) = S$.

5. (10 points) Determine whether or not the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x, y, z) = (x + y + z, y + z, z)$$

is invertible.

6. (10 points) Define an equivalence relation on $\mathbb{R}$ by saying $x \sim y$ if $x - y \in \mathbb{Q}$. Determine, with justification, whether each equivalence class is countable or uncountable, and whether the set of equivalence classes is countable or uncountable.

7. (10 points) A sequence $(r_1, r_2, r_3, \ldots)$ of rational numbers is *eventually constant* if there exists $r \in \mathbb{Q}$ and $N \in \mathbb{N}$ such that $r_n = r$ for all $n > N$. (In other words, all terms beyond some point are the same.) Show that the set of sequences of rational numbers which are eventually constant is countable.

8. (10 points) Show that the set $\mathbb{Q}^\infty$ of *all* sequences $(r_1, r_2, r_3, \ldots)$ of rational numbers is uncountable by showing directly that given any infinite list of elements of $\mathbb{Q}^\infty$, there always exists an element of $\mathbb{Q}^\infty$ not included in that list. (Or in other words, given any function $\mathbb{N} \to \mathbb{Q}^\infty$, there exists an element of $\mathbb{Q}^\infty$ not included in its image.)