Math 300: Midterm 1 Solutions
Northwestern University, Spring 2017

1. Give an example of each of the following with brief justification.

(a) Sets S, A, B such that $S \subseteq A \cup B$ but $S \not\subseteq A$ and $S \not\subseteq B$.

(b) A subset A of $\mathbb{R}$ such that $(\mathbb{R} \times \mathbb{R}) - (A \times A) \neq (\mathbb{R} - A) \times (\mathbb{R} - A)$.

Solution. (a) Take A to be the interval $(-\infty, 0]$, B to be the interval $[0, \infty)$, and S the interval $[-1, 1]$. Then $A \cup B = \mathbb{R}$ so $S \subseteq A \cup B$ but S is neither a subset of A nor B individually.

(b) Take A to be the interval $[-1, 1]$. Then the point $(0, 2)$ is not in $A \times A$ since $2 \notin [-1, 1]$, so $(0, 2)$ is in $(\mathbb{R} \times \mathbb{R}) - (A \times A)$, but $(0, 2)$ is not in $(\mathbb{R} - A) \times (\mathbb{R} - A)$ since $0 \notin \mathbb{R} - A$. $\square$

2. Suppose A and B are the following sets:

$$A = \{n \in \mathbb{Z} \mid n = 6k - 4 \text{ for some } k \in \mathbb{Z}\} \text{ and}$$

$$B = \{n \in \mathbb{Z} \mid n = 3k + 2 \text{ for some } k \in \mathbb{Z}\}.$$

Show that $A \subseteq B$ and $B \not\subseteq A$.

Proof. Let $n \in A$. Then there exists $k \in \mathbb{Z}$ such that $n = 6k - 4$. Rewriting this as

$$n = 6k - 4 = 3(2k - 2) + 2$$

shows there exists $\ell \in \mathbb{Z}$, namely $\ell = 2k - 2$, such that $n = 3\ell + 2$, so $n \in B$. Hence $A \subseteq B$. (To figure out which ℓ we need, set up $6k - 4 = 3\ell + 2$ and solve for ℓ in terms of k.)

Now, note that $5 \in B$ since we can write 5 as $5 = 3(1) + 2$. However, in order for 5 to be in A we would need the existence of $k \in \mathbb{Z}$ such that $5 = 6k - 4$, but only $k = \frac{3}{2}$ satisfies this property and $\frac{3}{2} \notin \mathbb{Z}$, so no such $k \in \mathbb{Z}$ exists. Hence $5 \notin A$, so $B \not\subseteq A$. (To be clear, $B \subseteq A$ means "if $x \in B$, then $x \in A$", so after negating wee that $B \not\subseteq A$ means "there exists $x \in B$ such that $x \notin A$".) $\square$

3. (a) Suppose $x \in \mathbb{R}$. Show that if $1 - r \leq x$ for all $r > 0$, then $1 \leq x$.

(b) Determine the following intersection, and prove that your answer is correct.

$$\bigcap_{r>0} (1 - r, 3].$$

To be clear, we are considering intervals $(1 - r, 3] = \{x \in \mathbb{R} \mid 1 - r < x \leq 3\}$ as r ranges over all positive real numbers.

Proof. (a) We instead prove the contrapositive: if $1 > x$, then there exists $r > 0$ such that $1 - r > x$. (To see what to do, rewrite the desired inequality as $1 - x > r$, so we need a positive r satisfying this inequality.) Suppose $1 > x$ and set $r = \frac{1}{2}(1 - x)$, which is positive since $1 - x > 0$. Then

$$1 - x > \frac{1}{2}(1 - x) = r, \text{ so } 1 - r > x$$

as desired. Thus the original implication is true.

(b) We claim that

$$\bigcap_{r>0} (1 - r, 3] = [1, 3].$$

Indeed, let $x \in [1,3]$. Then for any $r > 0$ we have

$$1 - r < 1 \le x \le 3,$$

so $x \in (1-r, 3]$ for all $r > 0$. Hence $x \in \bigcap_{r>0}(1-r,3]$, so $\bigcap_{r>0}(1-r,3] \supseteq [1,3]$.

Now let $x \in \bigcap_{r>0}(1-r,3]$. Then $x \in (1-r,3]$ for all $r > 0$. In particular, this gives that $x \le 3$, and since $1 - r < x$ for all $r > 0$, the result of part (a) gives that $1 \le x$. Thus $1 \le x \le 3$, so $x \in [1,3]$. Hence $\bigcap_{r>0}(1-r,3] \subseteq [1,3]$, so equality holds as claimed. □

4. Suppose A, B, C are sets. Show that

$$(A \cap B) - C = (A \cap B) - (A \cap C).$$

Proof. Let $x \in (A \cap B) - C$. Then $x \in A \cap B$ and $x \notin C$. Since $x \notin C$, $x \notin A \cap C$. (This comes from negating the definitions of $x \in A \cap C$: $x \in A \cap C$ means $x \in A$ and $x \in C$, so $x \notin A$ means $x \notin A$ or $x \notin C$.) Thus $x \in A \cap B$ and $x \notin A \cap C$, so $x \in (A \cap B) - (A \cap C)$. Hence $(A \cap B) - C \subseteq (A \cap B) - (A \cap C)$.

Conversely let $x \in (A \cap B) - (A \cap C)$. Then $x \in A \cap B$ and $x \notin A \cap C$. Since $x \notin A \cap C$, $x \notin A$ or $x \notin C$. But $x \in A \cap B$ gives that $x \in A$, so it must be the case that $x \notin C$. Hence $x \in A \cap B$ and $x \notin C$, so $x \in (A \cap B) - C$. Thus $(A \cap B) - C \supseteq (A \cap B) - (A \cap C)$, so equality holds. □

5. Suppose $n \in \mathbb{Z}$. Show that n is divisible by 10 if and only if n is divisible by both 2 and 5. (Recall that to say n is divisible by $k \in \mathbb{Z}$ means there exists $\ell \in \mathbb{Z}$ such that $n = k\ell$.)

It is NOT enough to say something along the lines of "if n has 2 and 5 as factors, then it must have $2 \cdot 5 = 10$ as a factor as well since 2 and 5 have no common factors apart from ± 1" without proof. Ask if you're unsure about whether you can take some fact for granted.

Proof. Suppose n is divisible by 10. Then there exists $k \in \mathbb{Z}$ such that $n = 10k$. This gives

$$n = 2(5k) \text{ and } n = 5(2k),$$

which show that n is divisible by 2 and 5 respectively.

Conversely suppose n is divisible by 2 and 5. Since n is divisible by 5, there exists $k \in \mathbb{Z}$ such that $n = 5k$. Now, if k were odd, $5k = n$ would be odd, in which case n would not be divisible by 2. Since n is divisible by 2, k must thus be even. Hence $k = 2\ell$ for some $\ell \in \mathbb{Z}$, so

$$n = 5k = 5(2\ell) = 10\ell,$$

showing that n is divisible by 10 as claimed. □