Math 300: Midterm 2

Northwestern University, Winter 2019

Name: ______________________________

1. (10 points) Give an example of each of the following with brief justification.

(a) An invertible function $f : \mathbb{Z} \to \mathbb{Z}$ which is not the identity function.

(b) A subset X of $\mathbb{R}$ such that $f^{-1}(f(X)) \neq X$ where $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = x^2$.

Problem	Score
1	
2	
3	
4	
5	
Total	

2. (10 points) Suppose $x_1 < 2$ and define the numbers x_n for $n > 1$ by setting

$$x_{n+1} = \frac{4x_n + 2}{5} \text{ for } n \geq 1.$$

Show that 2 is an upper bound of the set $\{x_n \mid n \in \mathbb{N}\}$ containing these numbers.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 3x+1 & \text{if } x > -1 \\ e^x & \text{if } x \leq -1 \end{cases}$$

(a) (7 points) Determine, with proof, the image $f([-2,1])$ of the interval $[-2,1]$ under f. (You can draw a picture of the graph of f to come up with a guess as to what the answer should be, but the picture alone does not constitute a proof.)

(b) (3 points) Also, determine what the preimage $f^{-1}([1/3,1])$ of the interval $[1/3,1]$ is, but you do not have to prove that your answer is correct in this case.

4. (10 points) Suppose $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f : A \to C$ is injective. Show that f is injective. (We did this in class, but of course you cannot simply quote this exact result from class.)

5. Define a relation $\sim$ on $\mathbb{R}^2$ by setting

$$(x, y) \sim (a, b) \text{ if } y - b = 3(x - a)$$

(a) (6 points) Show that $\sim$ is an equivalence relation.

(b) (4 points) For a fixed point $(x, y) \in \mathbb{R}^2$ find the element in the equivalence class $[(x, y)]$ of (x, y) which is on the x-axis. (You can take for granted the fact that each equivalence class intersects the x-axis in exactly one point.)