Math 320-3: Final Exam Practice
Northwestern University, Spring 2015

1. Give examples of each of the following.

(a) A non-orientable smooth surface in $\mathbb{R}^3$.

(b) A closed curve C such that $\int_C [-y/(x^2+y^2), x/(x^2+y^2)] \cdot \mathbf{T}\, ds \neq 0$.

(c) A smooth surface S whose unit normal vector at (x,y,z) is (x,y,z).

(d) A vector field $\mathbf{F}$ on $\mathbb{R}^2$ such that $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \pi ab$ where C is the ellipse $x^2/a^2 + y^2/b^2 = 1$ oriented counterclockwise.

(e) A vector field $\mathbf{F}$ on R^3 such that $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \operatorname{Vol} E$ for any closed smooth surface S, where E is the region enclosed by S.

Solution. (a) The Möbius strip is an example. Its parametric equations are given in the book or in the lecture notes, but I would NOT expect you to be able to recall those on the final.

(b) The unit circle centered at the origin oriented counterclockwise works. The computation is in the solution to the second problem on the final homework.

(c) The unit sphere centered at the origin is the only surface with this property. The point is that for spheres it is always true the outward pointing normal vector at a point (x,y,z) points in the same direction as the vector from $(0,0,0)$ to (x,y,z), and to have (x,y,z) itself have length 1 requires that we be on the unit sphere.

(d) The field $\mathbf{F} = (P,Q) = (0,x)$ is one example. By Green's Theorem, if D is the region enclosed by C we have

$$\int_C (0,x) \cdot \mathbf{T}\, ds = \iint_E (Q_x - P_y)\, dA = \iint_E dA = \text{area of } D = \pi ab.$$

Any field which has $Q_x - P_y = 1$ will work.

(e) The field $\mathbf{F} = (x,0,0)$ works. By Gauss's Theorem we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E dV = \operatorname{Vol} E.$$

Any field with divergence 1 will work. □

2. Wade, 13.1.11ab. Let C be a smooth C^2 arc with parametrization $(\phi, [a,b])$ and arc length L, and suppose that $s = \ell(t)$ is given by

$$\ell(t) = \int_a^t \|\phi'(u)\|\, du.$$

Let $(\psi, [0,L])$ be the parametrization of C defined by

$$\psi(s) = (\phi \circ \ell^{-1})(s).$$

This is what the book calls the *natural parametrization* of C, but is often also called the *parametrization with respect to arc length*.

(a) Prove that $\|\psi'(s)\| = 1$ for all s and the arc length of a subcurve $(\psi, [c,d])$ of C is $d - c$.

(b) Show that $\psi'(s)$ and $\psi''(s)$ are orthogonal for each $s \in [0,L]$.

Proof. (a) By the chain rule we have

$$\psi'(s) = (\phi \circ \ell^{-1})'(s) = \phi'(\ell^{-1}(s))\frac{d\ell^{-1}}{ds}(s).$$

By the Inverse Function Theorem, the derivative of ℓ^{-1} is given by

$$\frac{d\ell^{-1}}{ds}(s) = \frac{1}{\ell'(\ell^{-1}(s))},$$

which is the statement $D(\ell^{-1}) = (D\ell)^{-1}$ only that in this case these Jacobian matrices are 1×1. By the Fundamental Theorem of Calculus, $\ell'(t) = \|\phi'(t)\|$ for all t, so all together we get

$$\psi'(s) = \phi'(\ell^{-1}(s))\frac{d\ell^{-1}}{ds}(s) = \phi'(\ell^{-1}(s))\frac{1}{\ell'(\ell^{-1}(s))} = \phi'(\ell^{-1}(s))\frac{1}{\|\phi'(\ell^{-1}(s))\|}.$$

Taking norms gives $\|\psi'(s)\| = 1$ for all s as claimed. The arc length of the subcurve $(\psi, [c, d])$ is thus

$$\int_c^d \left\|\psi'(s)\right\| \, ds = \int_c^d \, ds = d - c.$$

(b) Since $\|\psi'(s)\| = \sqrt{\psi'(s) \cdot \psi'(s)} = 1$ for all s, we have that $\psi'(s) \cdot \psi'(s) = 1$ for all s as well. Thus taking derivatives with respect to s gives

$$\psi''(s) \cdot \psi'(s) + \psi'(s) \cdot \psi''(s) = 0, \text{ so } \psi'(s) \cdot \psi''(s) = 0$$

for all s as claimed. □

3. Wade, 13.2.5. Let (ϕ, I) be a smooth parametrization of some arc and τ be a C^1 function, 1-1 from J onto I, which satisfies $\tau'(u) > 0$ for all but finitely many $u \in J$. If $\psi = \phi \circ \tau$, prove that

$$\int_I \mathbf{F}(\phi(t)) \cdot \phi'(t)\, dt = \int_J \mathbf{F}(\psi(u)) \cdot \psi'(u)\, du$$

for any continuous $\mathbf{F} : \psi(I) \to \mathbb{R}^m$.

Proof. We denote the parameter in J by u and the one in I by t, so that $t = \tau(u)$. By the change of variables formula we have

$$\int_{I=\tau(J)} \mathbf{F}(\phi(t)) \cdot \phi'(t)\, dt = \int_J \mathbf{F}(\phi(\tau(u))) \cdot \phi'(\tau(u))|\tau'(u)|\, du.$$

For all but finitely many $u \in J$ we have $|\tau'(u)| = \tau'(u)$, so the integrand on the right side is the same as

$$\mathbf{F}(\phi(\tau(u))) \cdot \phi'(\tau(u))\tau'(u)$$

for all but finitely many $u \in J$. A finite set in J has measure zero, so we conclude that the integral on the right has the same value as the integral of the function above where we use $\tau'(u)$ instead of $|\tau'(u)|$:

$$\int_J \mathbf{F}(\phi(\tau(u))) \cdot \phi'(\tau(u))|\tau'(u)|\, du = \int_J \mathbf{F}(\phi(\tau(u))) \cdot \phi'(\tau(u))\tau'(u)\, du.$$

Since $\psi(u) = \phi(\tau(u))$, by the chain rule $\psi'(u) = \phi'(\tau(u))\tau'(u)$ so

$$\int_J \mathbf{F}(\phi(\tau(u))) \cdot \phi'(\tau(u))\tau'(u)\, du = \int_J \mathbf{F}(\psi(u)) \cdot \psi'(u)\, du.$$

Hence putting it all together we get

$$\int_I \mathbf{F}(\phi(t)) \cdot \phi'(t)\, dt = \int_J \mathbf{F}(\psi(u)) \cdot \psi'(u)\, du$$

as claimed. (The point is that the condition $\tau'(u) > 0$ for all but finitely many u implies that ϕ and ψ induce the same orientation at all but finitely many points, so they should give the same value for the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ in question.) $\square$

4. Wade, 13.3.5a. Suppose that $\psi(B)$ and $\phi(E)$ are C^p surfaces and that $\psi = \phi \circ \tau$, where τ is a C^1 function from B onto E. If (ψ, B) and (ϕ, E) are smooth and τ is 1-1 with $\det D\tau \neq 0$ on B, prove that

$$\iint_E g(\phi(u,v)) \, \|N_\phi(u,v)\| \; d(u,v) = \iint_B g(\psi(s,t)) \, \|N_\psi(s,t)\| \; d(s,t)$$

for all continuous $g : \phi(E) \to \mathbb{R}$.

Proof. To make the notation clear, (s,t) are the parameters in B and (u,v) the parameters in E, so $(u,v) = \tau(s,t)$. By change of variables we have

$$\iint_{E=\tau(B)} g(\phi(u,v)) \, \|N_\phi(u,v)\| \; d(u,v) = \iint_B g(\phi(\tau(s,t))) \, \|N_\phi(\tau(s,t))\| \, |\det D\tau(s,t)| \, d(s,t).$$

Recall that the normal vectors determined by ϕ and ψ are related by

$$N_\psi(s,t) = (\det D\tau(s,t)) N_\phi(\tau(s,t)), \text{ so } \|N_\psi(s,t)\| = |\det D\tau(s,t)| \, \|N_\phi(\tau(s,t))\| \, .$$

Thus

$$\iint_B g(\phi(\tau(s,t))) \, \|N_\phi(\tau(s,t))\| \, |\det D\tau(s,t)| \, d(s,t) = \iint_B g(\psi(s,t)) \, \|N_\psi(s,t)\| \; d(s,t),$$

so

$$\iint_E g(\phi(u,v)) \, \|N_\phi(u,v)\| \; d(u,v) = \iint_B g(\psi(s,t)) \, \|N_\psi(s,t)\| \; d(s,t)$$

as claimed. $\square$

5. Wade, 13.3.6. Suppose that $f : B_3(0,0) \to \mathbb{R}$ is differentiable with $\|\nabla f(x,y)\| \leq 1$ for all $(x,y) \in B_3(0,0)$. Prove that if S is the paraboloid $2z = x^2 + y^2, 0 \leq z \leq 4$, then

$$\iint_S |f(x,y) - f(0,0)| \, d\sigma \leq 40\pi.$$

Proof. (I can't get the bound 40π the book claims holds; the best I can get is 50π. So, I'm going to assume the book made a mistake and give a proof that 50π works instead. Please let me know if you're able to show that 40π works as well.)

The largest distance to the z-axis a point (x,y,z) on the paraboloid can attain is at the top where $z = 4$, in which case we have

$$8 = x^2 + y^2, \text{ so distance to } z\text{-axis} = \sqrt{x^2 + y^2} = \sqrt{8}.$$

This implies that for any $(x,y,z) \in S$, the point (x,y) in the xy-plane is in the ball of radius $\sqrt{8}$ around $(0,0)$, and so in particular is also in $B_3(0,0)$. Since any ball is convex, by the Mean Value

Theorem for any such point there exists (c,d) on the line segment between $(0,0)$ and (x,y) such that

$$f(x,y)-f(0,0)=\nabla f(c,d)[(x,y)-(0,0)], \text{ so } |f(x,y)-f(0,0)|=\|\nabla f(c,d)\|\,\|(x,y)\|\leq\|(x,y)\|$$

by the assumption given on $\|\nabla f\|$. As derived above, for any $(x,y,z)\in S$ we have $\|(x,y)\|\leq\sqrt{8}$, so we conclude that

$$|f(x,y)-f(0,0)|\leq\sqrt{8} \text{ for any } (x,y,z)\in S.$$

Thus

$$\iint_S |f(x,y)-f(0,0)|\,d\sigma\leq\iint_S\sqrt{8}\,d\sigma.$$

We now compute this final surface integral. (Everything up to this point is fair game for the final. I mentioned in class that I wouldn't ask you to compute surface integrals explicitly on the final, so the computation which follows won't show up. However, note that the answer is supposed to be $\sqrt{8}$ times the surface area of S, so if I gave you the value of the surface area of S in the problem statement—thereby bypassing the following computation—you should be able to finish it off. In other words, you should be able to recognize that $\iint_S 1\,d\sigma$ gives the surface area of S.) We parametrize S using

$$\phi(r,\theta)=(r\cos\theta,r\sin\theta,\frac{1}{2}r^2) \text{ for } (r,\theta)\in[0,\sqrt{8}]\times[0,2\pi].$$

Then

$$\phi_r\times\phi_\theta=(\cos\theta,\sin\theta,r)\times(-r\sin\theta,r\cos\theta,0)=(-r^2\cos\theta,-r^2\sin\theta,r),$$

so $\|\phi_r\times\phi_\theta\|=r\sqrt{1+r^2}$. Hence

$$\begin{aligned}\iint_S\sqrt{8}\,d\sigma&=\iint_{[0,\sqrt{8}]\times[0,2\pi]}\sqrt{8}\,\|\phi_r\times\phi_\theta\|\;d(r,\theta)\\&=\int_0^{2\pi}\int_0^{\sqrt{8}}\sqrt{8}\,r\sqrt{1+r^2}\,dr\,d\theta\\&=\int_0^{2\pi}\frac{\sqrt{8}}{3}(1+r^2)^{3/2}\Big|_0^{\sqrt{8}}\,d\theta\\&=\int_0^{2\pi}\frac{26}{3}\sqrt{8}\,d\theta\\&=\frac{52\pi\sqrt{8}}{3}.\end{aligned}$$

Since $\frac{52\pi\sqrt{9}}{3}<50\pi$ (it is approximately 49π), the claim follows. ∎

6. Wade, 13.4.7a. Suppose that S is a smooth surface. Show that there exist (finitely many) smooth parametrizations (ϕ_j,E_j) of portions of S such that $S=\bigcup_{j=1}^N\phi_j(E_j)$.

Proof. To clarify what this is saying, note that the book defines a smooth surface to be one where for each point $p\in S$ there exists a parametrization (ϕ,E) of S such $N_\phi(p)\neq\mathbf{0}$. However, for this parametrization there is no guarantee that $N_\phi\neq\mathbf{0}$ at points *other* than p. In other words, the parametrization needed to check smoothness might vary from point to point so there does not necessarily exist a *single* parametrization which is smooth (i.e. gives non-zero normal vectors) at all points at once. This is too tricky for the final in that it's not so clear what is actually required, in particular the use of compactness at the end, but fun to go through nonetheless!

Fix $p \in S$. Since S is smooth, there exists a parametrization (ϕ_p, D_p) which is smooth at p, meaning that $N_{\phi_p}(p) \neq \mathbf{0}$. Since S is at least a C^1 surface (it might be C^p for $p > 1$ as well), the assignment $q \mapsto N_{\phi_p}(q)$ is continuous, so not only is $N_{\phi_p}(p)$ nonzero at p, it must be nonzero in some neighborhood of p as well: in other words, there exists an open subset E_p of D_p on which N_{ϕ_p} is nonzero at all points, so the parametrization ϕ_p restricted to E_p gives a smooth parametrization (ϕ_p, E_p) for the portion of S given by the image $\phi_p(E_p)$.

Each of the images $\phi_p(E_p)$ is open in S (since C^1 one-to-one functions with invertible Jacobians–which parametrizatons are—send open sets to open sets), and so the collection of such images gives an open cover of S. Since S is compact (surfaces are always closed and bounded in $\mathbb{R}^3$), this collection has a finite subcover, say

$$\phi_1(E_1), \ldots, \phi_N(E_N).$$

Then each parametrization (ϕ_i, E_i) of a portion of S is smooth and $S = \bigcup_{j=1}^N \phi_j(E_j)$ as required. $\square$

7. Wade, 13.5.10e. Suppose that V is open and nonempty in $\mathbb{R}^2$, that u is C^2 on V, and that u is continuous on $\overline{V}$. Prove that u is harmonic on V if and only if

$$\int_{\partial E} (-u_y, u_x) \cdot \mathbf{T}\, ds = 0$$

for all two-dimensional regions $E \subseteq V$ which satisfy the hypotheses of Green's Theorem.

Check the rest of this problem in the book for a reminder as to what some of these terms mean. Also, note that I'm writing the line integral in question in vector form as opposed to in differential form as the book does, only because we didn't really talk about line integrals written in differential form until the final day of class, which I said would not be on the final.

Proof. Suppose that u is harmonic on V and let $E \subseteq V$ be a two-dimensional region satisfying the hypotheses of Green's Theorem. Then for the C^1 vector field $\mathbf{F} = (P, Q) = (-u_y, u_x)$, Green's Theorem gives

$$\int_{\partial E} (-u_y, u_x) \cdot \mathbf{T}\, ds = \iint_E (Q_x - P_y)\, dA = \iint_E (u_{xx} + u_{yy})\, dA.$$

Since u is harmonic on V, $u_{xx} + u_{yy} = 0$ so this integral on the right is zero and hence the line integral on the left is zero as claimed.

Conversely suppose that $\int_{\partial E}(-u_y, u_x)\cdot \mathbf{T}\, ds = 0$ for any two-dimensional region $E \subseteq V$ satisfying the hypotheses of Green's Theorem. Then the Green's Theorem application above gives

$$\iint_E (u_{xx} + u_{yy})\, dA = 0$$

for any such two-dimensional region $E \subseteq V$. If $u_{xx} + u_{yy} \neq 0$ at some point, then continuity of this expression (which follows from the C^2 condition on u) would imply that $u_{xx} + u_{yy}$ is either strictly positive or strictly negative on some open ball in V around that point, which would give a positive or negative value for the integral of $u_{xx} + u_{yy}$ over that ball. Hence the fact that all such integrals are zero implies that $u_{xx} + u_{yy}$ must identically zero throughout V, so that u is harmonic on V. $\square$

8. Wade, 13.6.7a. Let S be an orientable surface with unit normal $\mathbf{n}$ and nonempty boundary ∂S which satisfies the hypotheses of Stokes' Theorem. Suppose that $\mathbf{F} : S \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is C^1, that ∂S is smooth, and that $\mathbf{T}$ is the unit tangent vector on ∂S induced by $\mathbf{n}$. If the angle between $T(\mathbf{x}_0)$ and $\mathbf{F}(x_0)$ is never obtuse (i.e. greater than 90°) for any $\mathbf{x}_0 \in \partial S$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$, prove that $\mathbf{T}(\mathbf{x}_0)$ and $\mathbf{F}(\mathbf{x}_0)$ are orthogonal for all $\mathbf{x}_0 \in \partial S$.

Proof. By Stokes' Theorem we have

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

Now, using the fact that $\mathbf{F} \cdot \mathbf{T} = \|\mathbf{F}\|\, \|\mathbf{T}\| \cos\theta$ where θ is the angle between $\mathbf{F}$ and $\mathbf{T}$, the assumption that the angle between $\mathbf{F}$ and $\mathbf{T}$ is never obtuse implies $\mathbf{F} \cdot \mathbf{T} \geq 0$ at all points of ∂S. Thus since the integral of this continuous nonnegative dot product over ∂S is zero, we must have $\mathbf{F} \cdot \mathbf{T} = 0$ at all points of ∂S, showing that $\mathbf{F}(\mathbf{x}_0)$ and $\mathbf{T}(\mathbf{x}_0)$ are orthogonal for all $\mathbf{x}_0 \in \partial S$ as desired. □

9. Wade, 13.6.8. Suppose that E is a two-dimensional region such that if $(x, y) \in E$, then the line segments from $(0, 0)$ to $(x, 0)$ and form $(x, 0)$ to (x, y) are both subsets of E. If $\mathbf{F} : E \to \mathbb{R}^2$ is C^1, prove that the following three statements are equivalent.

(a) $\mathbf{F} = \nabla f$ on E for some $f : E \to \mathbb{R}$.

(b) $\mathbf{F} = (P, Q)$ satisfies $Q_x = P_y$. (The book call this condition being *exact*, which is a nonstandard usage of that term; most people would refer to this as saying that $\mathbf{F}$ is *closed*, which is related to the differential form approach to vector fields I outlined on the last day of class. In this setting, being *exact* is precisely the condition in part (a).)

(c) $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$ for all piecewise smooth curves $C = \partial\Omega$ oriented counterclockwise, where Ω is a two-dimensional region which satisfies the hypotheses of Green's Theorem and $\Omega \subseteq E$.

Note that I changed the original version of the problem I had to simply be the book's version instead; my version was just a little too tricky and the book's version already contains the key concepts anyway. I'll just point out, as my version suggested, that the equivalences given here hold for other types of regions more generally, such as open connected regions and so-called simply-connected ones—it's just that proving the equivalences in these settings requires more work that's not worth going through for the purposes of the final.

Proof. (a $\implies$ b) Suppose that $\mathbf{F} = (P, Q) = \nabla f$ on E for some $f : E \to \mathbb{R}$, which is necessarily C^2 since $\mathbf{F}$ is C^1. Then $P = f_x$ and $Q = f_y$, so $Q_x = f_{yx} = f_{xy} = P_y$ by Clairaut's Theorem.

(b $\implies$ c) Suppose that $\mathbf{F} = (P, Q)$ satisfies $Q_x = P_y$. Then for any two-dimensional region $\Omega \subseteq E$ with piecewise smooth boundary, we have by Green's Theorem:

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_\Omega (Q_x - P_y)\, dA = \iint_\Omega 0\, dA = 0$$

as claimed in part (c).

(c $\implies$ a) We essentially did this part when showing that "path-independence implies conservative" on May 22nd in class. The idea is to define the function $f : E \to \mathbb{R}$ via a line integral and check that the gradient of the resulting f is indeed $\mathbf{F}$. We define

$$f(x, y) = \int_{C(x,y)} \mathbf{F} \cdot \mathbf{T}\, ds \text{ for } (x, y) \in E$$

where $C(x, y)$ is the path from $(0, 0)$ to (x, y) consisting of the horizontal line segment from $(0, 0)$ to $(x, 0)$ followed by the vertical line segment from $(x, 0)$ to (x, y); both of these segments lie in E

by the given assumptions. Now, the proof that $f_y = Q$ is the same as the one given in the lecture notes for May 22nd where we use $(x_0, y_0) = (0, 0)$, so we'll omit that here. Note that proof we gave on the day for $f_x = P$ does not work here since we are not guaranteed that the line segments we used there are actually contained in our region E. So, instead we use another idea given in the lecture notes for that day.

After parametrizing the given segments, we get

$$f(x, y) = \int_{x_0}^{x} P(t, y_0)\, dt + \int_{y_0}^{y} Q(x, t)\, dt$$

as computed on May 22nd. Differentiating with respect to x gives

$$f_x(x, y) = P(x, y_0) + \int_{y_0}^{y} Q_x(x, t)\, dt.$$

Now, part (c) actually implies part (b), so we know that $Q_x = P_y$. (The proof of this is to run the proof of b $\implies$ c in reverse: we have that for any Ω, $\iint_\Omega (Q_x - P_y)\, dA = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{T}\, ds = 0$ by Green's Theorem, which since $Q_x - P_y$ is continuous implies that it must be zero.) Thus we can write the expression above as:

$$f_x(x, y) = P(x, y_0) + \int_{y_0}^{y} P_y(x, t)\, dt,$$

which by the Fundamental Theorem of Calculus gives

$$f_x(x, y) = P(x, y_0) + P(x, y) - P(x, y_0) = P(x, y).$$

Thus $\nabla f = (f_x, f_y) = (P, Q) = \mathbf{F}$ as required. □

10. Wade, 13.6.11. Let $\mathbf{F}$ be C^1 and exact (according to the book's definition in Exercise 13.6.8) on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

(a) Suppose that C_1 and C_2 are disjoint smooth simple curves, oriented in the counterclockwise direction, and that E is a two-dimensional region whose topological boundary ∂E is the union of C_1 and C_2. If $(0, 0) \notin E$, prove that

$$\int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds.$$

(b) Suppose that E is a two-dimensional region which satisfies $(0, 0) \in E^o$. If ∂E is a smooth simple curve oriented in the counterclockwise direction and

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right),$$

compute $\int_{\partial E} \mathbf{F} \cdot \mathbf{T}\, ds$.

(c) State and prove an analogue of part a) for fields $\mathbf{F}$ on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, three-dimensional regions, and smooth surfaces.

Proof. (a) Since $(0, 0) \in E$, $\mathbf{F}$ is C^1 and so Green's Theorem implies that

$$\int_{\partial E} \mathbf{F} \cdot \mathbf{T}\, ds = \iint_E (Q_x - P_y)\, dA = \iint_E 0\, dA = 0$$

since F is exact. Supposing that C_2 is the piece of the boundary ∂E which lies further away from the origin (i.e. the "outer" piece) and C_1 the piece lying closer to the origin (i.e. the "inner" piece), the positive orientation on ∂E has C_2 oriented counterclockwise but C_1 oriented clockwise. Thus

$$0 = \int_{\partial E} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{-C_1} \mathbf{F} \cdot\, ds,$$

where $-C_1$ denotes C_1 with the clockwise orientation. But the integral over $-C_1$ is negative the integral over C_1, so

$$0 = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds - \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds,$$

from which the claimed equality follows.

(b) For this field $\mathbf{F} = (P, Q)$, we have

$$Q_x = \frac{y^2 - x^2}{(x^2 + y^2)} = P_y,$$

so $\mathbf{F}$ is exact. Since E^o is open in $\mathbb{R}^2$, there exists $r > 0$ such that $B_r(0,0) \subseteq E^o \subseteq E$. The boundary of the region obtained by removing $B_r(0,0)$ from E then has boundary which is the union of ∂E and $\partial B_r(0,0)$, so since $E \setminus B_r(0,0)$ no longer contains the origin, part (a) gives

$$\int_{\partial E} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\partial B_r(0,0)} \mathbf{F} \cdot \mathbf{T}\, ds$$

where $\partial B_r(0,0)$ has the counterclockwise orientation. The curve $\partial B_r(0,0)$ is the circle of radius r centered at the origin, so has parametric equations

$$\phi(t) = (r \cos t, r \sin t) \text{ for } 0 \leq t \leq 2\pi.$$

Then

$$\int_{\partial B_r(0,0)} \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{2\pi} \left(-\frac{\sin t}{r}, \frac{\cos t}{r} \right) \cdot (-r \sin t, r \cos t)\, dt = \int_0^{2\pi} dt = 2\pi,$$

so $\int_{\partial E} \mathbf{F} \cdot \mathbf{T}\, ds = 2\pi$ as well.

(c) The analog is the following. Let $\mathbf{F}$ be a C^1 vector field with zero divergence on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$. (The zero divergence property is the correct analog of $Q_x = P_y$.) Suppose that S_1 and S_2 are disjoint smooth surfaces with outward orientation and that E is a three-dimensional region whose topological boundary ∂E is the union of S_1 and S_2. If $(0,0,0) \neq E$, then

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

The proof is as follows. By Gauss's Theorem, we have

$$\iint_{\partial E} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 0\, dV = 0.$$

Supposing that S_1 lies further away from the origin than S_2, the positive orientation on ∂E induces the outward orientation on S_1 and the inner orientation on S_2, so

$$0 = \iint_{\partial E} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, d\sigma + \iint_{-S_2} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Flipping orientation changes the sign of a surface integral, so

$$0 = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, d\sigma - \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, d\sigma,$$

and the claimed equality follows. $\square$