Math 320-2: Final Exam Practice Solutions
Northwestern University, Winter 2015

1. Give an example of each of the following. No justification is needed.

(a) A closed and bounded subset of $C[0,1]$ which is not compact.
(b) An unbounded subset of $\mathbb{R}^3$ which is not connected.
(c) A continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is not uniformly continuous.
(d) A non-constant function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}$ is open, which has derivative zero.
(e) A compact subset of $C[0,1]$.

Solution. (a) The closed ball of radius 1 around the zero functions works. Indeed, this is bounded and closed since closed balls are always closed in arbitrary metric spaces. But it is not compact since the sequence (f_n) defined by $f_n(x) = x^n$ is in this ball but has no (uniformly) convergent subsequence.

(b) The subset consisting of the union of the horizontal lines $y = 0$ and $y = 1$ works, as do plenty of other things.

(c) The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x^2, 0)$ works, analogous to how the function $g(x) = x^2$ from $\mathbb{R} \to \mathbb{R}$ was not uniformly continuous.

(d) Take $U = (-1, 0) \cup (1, 2)$ and define $f : U \to \mathbb{R}$ by $f(x) = 0$ for $x \in (-1, 0)$ and $f(x) = 1$ for $x \in (1, 2)$. This is not constant but has zero derivative everywhere. The point here is that U is not connected.

(e) Any subset consisting of a single point, say $\{f\}$ where f is the function $f(x) = x$, is compact since finite sets are always compact. □

2. For subsets $A, B \in \mathbb{R}^2$, define the *distance* between A and B to be

$$d(A, B) := \inf\{d(p, q) \mid p \in A \text{ and } q \in B\}.$$

Show that if A and B are compact, then there exist $a \in A$ and $b \in B$ such that $d(A, B) = d(a, b)$, and that this is not necessarily true if A and B are merely closed.

We looked at this problem during the Warm-Up on the last day of class, where the idea came down to showing that the function $\mathbb{R}^4 \to \mathbb{R}$ which sends (x, y, z, w) to the distance between (x, y) and (z, w) in $\mathbb{R}^2$ was continuous. I said that I would include this problem on this set of practice problems, so here it is. However, I actually went ahead and wrote up the proof as part of the lecture notes. It's a little tricky to get everything right, so the continuity portion of this is too difficult for the final exam; the application of the Extreme Value Theorem and the asked for counterexample, however, are fair game.

Solution. As mentioned, you can find a proof in the lecture notes. As for the example showing that this is not necessarily true if A and B are merely closed, take A to be the piece of the curve $y = \frac{1}{x}$ in the first quadrant and B to be the piece of the curve $y = -\frac{1}{x}$ in the fourth quadrant. These are each closed in $\mathbb{R}^2$ and the distance between them as defined here is 0 since as $x \to \infty$ both curves get arbitrarily close to the x-axis, so the distance between a point on one curve and a point on the other gets closer and closer to 0. However, there do not exist points one the two curves for which this distance is exactly zero since the two curves do not intersect. □

3. Suppose that K is a subset of $\mathbb{R}^n$ with the property that every continuous function $f : K \to \mathbb{R}$ is bounded. Show that K is compact. Hint: Show that K is bounded by finding the right continuous function to apply our assumption to, and then show that if K was not closed there would exist a continuous function on K which was not bounded.

Proof. For a fixed $p \in K$, the function $f : K \to \mathbb{R}$ defined by $f(q) = d(p,q)$ is continuous as shown on Problem 4 of the final homework. Thus by our given assumption, this function is bounded, say by M. But this means that $d(p,q) \leq M$ for all $q \in K$, so $K \subseteq B_{M+1}(p)$ and hence K is bounded.

Now, if K is not closed there exists a sequence (p_n) in K which converges to some $p \notin K$. The function $g : K \to \mathbb{R}$ defined by $g(q) = \frac{1}{d(q,p)}$ is then continuous since the denominator is continuous and never zero on K. However, this function is unbounded since $f(p_n) = \frac{1}{d(p_n,p)} \to \infty$ since $d(p_n,p) \to 0$. This goes against our assumption on K, so K must be closed. Thus since K is closed and bounded in $\mathbb{R}^n$, it is compact as claimed by the Heine-Borel Theorem.

(It is actually true more generally that if K is *any* metric space, not necessarily assuming it is a subset of $\mathbb{R}^n$, with the property that any continuous function $K \to \mathbb{R}$ is bounded must be compact, although this is slightly harder to prove in general since then it is not enough to show only that K is bounded and closed inside of some larger space.) $\square$

4. Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and $\mathbb{Z}^2$-valued, then f is constant. (To say that f is $\mathbb{Z}^2$-valued means that $f(p) \in \mathbb{Z} \times \mathbb{Z}$ for all $p \in \mathbb{R}^2$, so that both coordinates of $f(p)$ are integers.)

Proof. Since $\mathbb{R}^2$ is connected and f is continuous, the image $f(\mathbb{R}^2)$ of f is connected and lies inside $\mathbb{Z}^2$. But the only connected subsets of $\mathbb{Z}^2$ are the empty set and the sets consisting of single points, so since $f(\mathbb{R}^2)$ is nonempty we must have $f(\mathbb{R}^2) = \{p\}$ for some $p \in Z^2$. Thus $f(x) = p$ for all $x \in \mathbb{R}^2$, so f is constant.

For completeness, here is a proof that the only nonempty connected subsets of $\mathbb{Z}^2$ are those consisting of single points. Suppose that $A \subseteq \mathbb{Z}^2$ is nonempty and connected, and let $p \in A$. Every subset of $\mathbb{Z}^2$ is open and closed in $\mathbb{Z}^2$, so $\{p\}$ is open and closed in $A \cap \mathbb{Z}^2 = A$. Since A is connected, the only subsets of A which are both open and closed are the empty set and A itself, so since $\{p\}$ is not empty we must have $A = \{p\}$, so that A consists of a single point as claimed. $\square$

5. Show that a function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is continuous if and only if the preimage $f^{-1}(A)$ of any closed subset A of Y is closed in X.

Proof 1. Suppose that f is continuous and that $A \subseteq Y$ is closed. Suppose that (p_n) is a sequence in $f^{-1}(A)$ which converges to some $p \in X$. Since f is continuous, $f(p_n) \to f(p)$. But each $f(p_n) \in A$ since $p_n \in f^{-1}(A)$, so $f(p_n)$ is a sequence in A which converges to $f(p) \in Y$, and since A is closed in Y we must have $f(p) \in A$. Thus $p \in f^{-1}(A)$, showing that $f^{-1}(A)$ is closed in X.

Conversely suppose that the preimage of any closed subset of Y is closed in X. Suppose that $p_n \to p$ in X. To show that f is continuous we must show that $f(p_n) \to f(p)$. If this was not true, there would exist $\epsilon > 0$ such that

$$\text{for any } N \in \mathbb{N}, \text{ there exists } n \geq N \text{ such that } d_Y(f(p_n,), f(p)) \geq \epsilon.$$

Applying this to increasing values of N gives a subsequence $(f(p_{n_k}))$ of $f((p_n))$ such that

$$d_Y(f(p_{n_k}), f(p)) \geq \epsilon \text{ for all } k.$$

Now, the set $A = \{q \in Y \mid d(q, f(p)) \geq \epsilon\}$ is closed in Y since it is the complement of the open ball $B_\epsilon(f(p))$, so its preimage $f^{-1}(A)$ is closed in X by our assumption. Since each $f(p_{n_k})$ is in A, each p_{n_k} is in $f^{-1}(A)$. But these p_{n_k} form a subsequence of (p_n) and so must converge to the same p as (p_n) does. Since the preimage $f^{-1}(A)$ is closed in X, we must thus have $p \in f^{-1}(A)$ meaning that $f(p) \in A$. But this is nonsense since it implies $d(f(p), f(p)) \geq \epsilon > 0$, so $f(p_n)$ must have converged to $f(p)$ to begin with, and therefore f is continuous. $\square$

Proof 2. (The previous proof is the more important one to digest since it emphasizes the relation between continuity and sequence convergence. This proof is just meant to illustrate the relation between the claimed result and the characterization of continuity we previously saw in terms of open sets. Both of these characterizations are good to know.)

First recall the following set equality, which holds for any $A \subseteq Y$:

$$f^{-1}(A^c) = f^{-1}(A)^c,$$

so that the preimage of a complement is the complement of the preimage. You can verify this by showing an element of one side is always an element of the other, but we'll skip this verification here. (You likely did this sort of thing in Math 300.)

This equality together with the fact that the complement of a closed set is open and the complement of an open set is closed, implies that the preimage image of a closed set is closed if and only if the preimage of an open set is open, which we have seen is equivalent to continuity. To flesh out the details, suppose that f is continuous and that A is closed in Y. Then A^c is open in Y, so since f is continuous the preimage $f^{-1}(A^c)$ is open in X. But this preimage is the same as the complement of $f^{-1}(A)$, and since this complement is open we have that $f^{-1}(A)$ is closed as desired.

Conversely, suppose that the preimage of any closed subset of Y is closed in X and let $U \subseteq Y$ be open. Then U^c is closed in Y so $f^{-1}(U^c)$ is closed in X. But this preimage is the same as the complement of $f^{-1}(U)$, so since this complement is closed X we have that $f^{-1}(U)$ is open in X. Thus the preimage of any open subset of Y is open in X, so f is continuous. $\square$

6. Wade, 9.5.1. Identify which of the following sets are compact and which are not. If E is not compact, find the smallest compact set H (if there is one) such that $E \subseteq H$.

(a) $\{1/k \mid k \in \mathbb{N}\} \cup \{0\}$

(b) $\{(x, y) \in \mathbb{R}^2 \mid a \leq x^2 + y^2 \leq b\}$ for real numbers $0 < a < b$

(c) $\{(x, y) \in \mathbb{R}^2 \mid y = \sin(1/x) \text{ for some } x \in (0, 1]\}$

(d) $\{(x, y) \in \mathbb{R}^2 \mid |xy| \leq 1\}$

Solution. (a) This is closed and bounded in $\mathbb{R}$, so it is compact. To be clear, it is closed in $\mathbb{R}$ since a sequence in this set is either eventually constant, or is one that converges to 0, which is included.

(b) This is closed and bounded in $\mathbb{R}^2$, so it is compact. It is bounded since it is contained in the ball of radius $b+1$ centered at the origin, and it is closed since it contains its boundary, which consists of the union of the circle of radius a with the circle of radius b centered at the origin.

(c) This is not compact since it is not closed in $\mathbb{R}^2$. For the sequence $x_n = \frac{1}{n\pi}$ in $(0, 1]$, we have

$$\sin \frac{1}{x_n} = \sin n\pi = 0,$$

so the points $\left(\frac{1}{n\pi}, 0\right)$ are in this set. However, these points converges to $(0, 0)$, which is not in this set and hence this set is not closed as claimed. The smallest compact subset of $\mathbb{R}^2$ which contains this set is its closure, which is

$$H = \{\text{the given set}\} \cup \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\},$$

or in other words what you get when you take the given set and include the points on the y-axis with y-coordinate between 0 and 1. This H is a famous example known as the "topologist's sine curve", and we'll look at it a bit more next quarter when talking about what it means for a set to be "path connected".

(d) This set set is not compact since it is not bounded. In particular, the points $(n, \frac{1}{n})$ are all in this set but have x-coordinates which get arbitrarily large. In this case, there is no compact subset H of $\mathbb{R}^2$ which contains this set, since if so the boundedness of that H would imply that this given set was bounded as well. $\square$

7. Suppose that $K_1, K_2, K_3, \dots$ are nonempty compact subsets of a metric space X such that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots .$$

Show that the intersection $\bigcap K_n$ of all of these is nonempty and that this is not necessarily true if we assume only that the K_n are closed. You can do this with either sequences or open covers, but the sequence approach is probably simpler.

Solution. Each each K_n is nonempty, for each n we can pick $p_n \in K_n$. Since all K_n are subsets of K_1, this gives a sequence (p_n) in K_1. Since K_1 is compact, this sequence has a convergent subsequence (p_{n_k}) which converges to some $p \in K_1$. We claim that this same p is actually in all K_n.

All terms of (p_{n_k}) except for the possibly the first (in the case when p_{n_1} is just p_1) are in K_2 since $K_{n_k} \subseteq K_2$ for $n_k \geq 2$. Since K_2 is compact, this sequence of p_{n_k}'s for $k \geq 2$ has a subsequence converging to some $q \in K_2$, but since this subsequence is in turn a subsequence of the original (p_{n_k}), this subsequence will converge to the same thing as (p_{n_k}) and hence we must have $q = p$, so $p \in K_2$.

Similarly, the subsequence of (p_{n_k}) where we start with p_{n_3} is a sequence in K_3, so since K_3 is compact this has a convergent subsequence in K_3, which must converge to the same p as does (p_{n_k}), so $p \in K_3$. Continuing in this manner, taking the subsequence of p_{n_k} where we start with a subsequent term at each step, will show that $p \in K_n$ for all n, so $p \in \bigcap K_n$ and hence this intersection is nonempty.

To show that this conclusion fails if only assume that the K_n are closed and not compact, take $K_n = [n, \infty)$ in $\mathbb{R}$. These sets are all closed in $\mathbb{R}$ but their intersection $\bigcap_n [n, \infty)$ is empty since the left endpoint n gets arbitrarily large, so fro any $x \in \mathbb{R}$ eventually n is large enough to exclude x from being in the intersection. $\square$

Solution using open covers. Here's a proof of the claim using the open cover approach, which is trickier to understand. I'm putting it here to be complete, but you should not expect something like this on the final.

Suppose for a contradiction that $\bigcap K_n = \emptyset$. Then the complement of this intersection must be all of X:

$$\left(\bigcap K_n\right)^c = \bigcup K_n^c = X.$$

Since each K_n is compact, each K_n is closed in X and hence each K_n^c is open. Thus the collection of complements $\{K_n^c\}$ forms an open cover of X and hence of K_1 as well. Since K_1 is compact this has a finite subcover, say:

$$K_1 \subseteq K_{n_1}^c \cup \cdots \cup K_{n_t}^c$$

where the indices n_i are arranged in increasing order. Since $K_{n_1} \supseteq K_{n_2} \supseteq \cdots \supseteq K_{n_t}$, we have

$$K_{n_1}^c \subseteq \cdots \subseteq K_{n_t}^c$$

so $K_{n_1}^c \cup \cdots \cup K_{n_t}^c = K_{n_t}^c$. Since K_{n_t+1} is nonempty there exists $p \in K_{n_t+1}$, and then $p \in K_{n_t}$ and $p \in K_1$ as well since K_{n_t+1} is contained in each of these. But since $K_1 \subseteq K_{n_t}^c$ we also have $p \in K_{n_t}^c$, which contradicts $p \in K_{n_t}$. Thus we must have had $\bigcap K_n \neq \emptyset$ to begin with. $\square$

8. Determine whether or not each of the following functions from $\mathbb{R}^2$ to $\mathbb{R}$ is continuous at $(0,0)$.

$$(a)\ f(x,y)=\begin{cases}\frac{3x^2y}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}\qquad (b)\ g(x,y)=\begin{cases}\frac{2xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Solution. (a) This is continuous at $(0,0)$. Indeed, note that if $(x,y)\neq(0,0)$ we have

$$|f(x,y)|=\left|\frac{3x^2y}{x^2+y^2}\right|\leq\frac{3(x^2+y^2)|y|}{x^2+y^2}=3|y|\leq 3\sqrt{x^2+y^2}.$$

Thus for $\epsilon>0$, setting $\delta=\frac{\epsilon}{3}$ gives

$$|f(x,y)-f(0,0)|=|f(x,y)|\leq 3\sqrt{x^2+y^2}<3\delta=\epsilon \text{ whenever } \sqrt{x^2+y^2}<\delta,$$

showing that f is continuous at $(0,0)$.

(b) This is not continuous at $(0,0)$. Consider the sequence $\left(\frac{1}{n},\frac{1}{n}\right)$ in $\mathbb{R}^2$. This converges to $(0,0)$, but

$$g\left(\frac{1}{n},\frac{1}{n}\right)=\frac{2/n^2}{2/n^2}=1$$

does not converge to $g(0,0)=0$. $\square$

9. Show that if E is a connected subset of a metric space X, then $\overline{E}$ is connected as well. The fact that if U is open in $\overline{E}$, then $U\cap E$ is open in E will be useful.

Proof. We prove the contrapositive: if $\overline{E}$ is not connected, then E is not connected. If $\overline{E}$ is not connected then there exist disjoint, nonempty open subsets U and V of $\overline{E}$ such that

$$\overline{E}=U\cup V.$$

Intersecting with E gives:

$$E=(U\cap E)\cup(V\cap E),$$

where each of $U\cap E$ and $V\cap E$ are open in E and are still disjoint since they sit inside of the disjoint sets U and V respectively. We claim that each of $U\cap E$ and $V\cap E$ are nonempty, which will show that E is not connected.

Since U and V are nonempty, there exist $p\in U$ and $q\in V$. Since $\overline{E}=U\cup V$, these p and q are limit points of E. Hence there exist sequences (p_n) and (q_n) in E such that $p_n\to p$ and $q_n\to q$. Since $p\in U$ and U is open in $\overline{E}$, there exists an open ball $B_r(p)\subseteq U$. Then for large enough n, $p_n\in B_r(p)$ since $p_n\to p$, and thus these p_n are in $U\cap E$. Similarly, there exists $B_s(q)\subseteq V$ and then for large enough n, $q_n\in B_s(q)$ since $q_n\to q$, so these q_n are in $V\cap E$. Hence $U\cap E$ and $V\cap E$ are each nonempty, so E is disconnected as claimed. $\square$

10. For a given $f\in C[2,5]$, define $Tf\in C[2,5]$ to be function whose value at $x\in[2,5]$ is

$$(Tf)(x)=3+\int_2^x(2xf(t)+e^t)\,dt.$$

Show that T is continuous. Hint: Show that there exists $K>0$ such that

$$d(Tf,Tg)\leq Kd(f,g) \text{ for all } f,g\in C[2,5]$$

where d is the sup metric on $C[2,5]$.

Proof. First, note that for any $x \in [2,5]$:

$$(Tf)(x)-(Tg)(x) = \left(3+\int_2^x (2xf(t)+e^t)\,dt\right)-\left(3+\int_2^x (2xg(t)+e^t)\,dt\right) = \int_2^x 2x(f(t)-g(t))\,dt.$$

Thus for $x \in [2,5]$, we have:

$$\begin{aligned}
|(Tf)(x)-(Tg)(x)| &= \left|\int_2^x 2x(f(t)-g(t))\,dt\right| \\
&\le \int_2^x 2|x||f(t)-g(t)|\,dt \\
&\le \int_2^x 10d(f,g)\,dt \\
&= 10d(f,g)(x-2) \\
&\le 30d(f,g).
\end{aligned}$$

Thus the number $30d(f,g)$ is an upper bound for the set of all expressions $|(Tf)(x)-(Tg)(x)|$ as x varies in $[2,5]$, so $d(Tf,Tg) \le 30d(f,g)$.

For any $\epsilon > 0$, set $\delta = \frac{\epsilon}{30}$. Then if $d(f,g) < \delta$, we have

$$d(Tf,Tg) \le 30d(f,g) < 30\delta = \epsilon,$$

so T is (uniformly) continuous as claimed. (Fun fact: by choosing an appropriate smaller interval $[2,k]$, we can guarantee that T is a contraction and hence will have a fixed-point on that smaller interval.) $\square$