Math 320-3: Final Exam Practice
Northwestern University, Spring 2015

1. Give examples of each of the following.

(a) A non-orientable smooth surface in $\mathbb{R}^3$.

(b) A closed curve C such that $\int_C [-y/(x^2+y^2), x/(x^2+y^2)] \cdot \mathbf{T}\, ds \neq 0$.

(c) A smooth surface S whose unit normal vector at (x, y, z) is (x, y, z).

(d) A vector field $\mathbf{F}$ on $\mathbb{R}^2$ such that $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \pi ab$ where C is the ellipse $x^2/a^2 + y^2/b^2 = 1$ oriented counterclockwise.

(e) A vector field $\mathbf{F}$ on R^3 such that $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \operatorname{Vol} E$ for any closed smooth surface S, where E is the region enclosed by S.

2. Wade, 13.1.11ab. Let C be a smooth C^2 arc with parametrization $(\phi, [a, b])$ and arc length L, and suppose that $s = \ell(t)$ is given by

$$\ell(t) = \int_a^t \|\phi'(u)\|\, du.$$

Let $(\psi, [0, L])$ be the parametrization of C defined by

$$\psi(s) = (\phi \circ \ell^{-1})(s).$$

This is what the book calls the *natural parametrization* of C, but is often also called the *parametrization with respect to arc length.*

(a) Prove that $\|\psi'(s)\| = 1$ for all s and the arc length of a subcurve $(\psi, [c, d])$ of C is $d - c$.

(b) Show that $\psi'(s)$ and $\psi''(s)$ are orthogonal for each $s \in [0, L]$.

3. Wade, 13.2.5. Let (ϕ, I) be a smooth parametrization of some arc and τ be a C^1 function, 1-1 from J onto I, which satisfies $\tau'(u) > 0$ for all but finitely many $u \in J$. If $\psi = \phi \circ \tau$, prove that

$$\int_I \mathbf{F}(\phi(t)) \cdot \phi'(t)\, dt = \int_J \mathbf{F}(\psi(u)) \cdot \psi'(u)\, du$$

for any continuous $\mathbf{F} : \psi(I) \to \mathbb{R}^m$.

4. Wade, 13.3.5a. Suppose that $\psi(B)$ and $\phi(E)$ are C^p surfaces and that $\psi = \phi \circ \tau$, where τ is a C^1 function from B onto E. If (ψ, B) and (ϕ, E) are smooth and τ is 1-1 with $\det D\tau \neq 0$ on B, prove that

$$\iint_E g(\phi(u, v))\, \|N_\phi(u, v)\|\ d(u, v) = \iint_B g(\psi(s, t))\, \|N_\psi(s, t)\|\ d(s, t)$$

for all continuous $g : \phi(E) \to \mathbb{R}$.

5. Wade, 13.3.6. Suppose that $f : B_3(0, 0) \to \mathbb{R}$ is differentiable with $\|\nabla f(x, y)\| \leq 1$ for all $(x, y) \in B_3(0, 0)$. Prove that if S is the paraboloid $2z = x^2 + y^2, 0 \leq z \leq 4$, then

$$\iint_S |f(x, y) - f(0, 0)|\, d\sigma \leq 40\pi.$$

6. Wade, 13.4.7a. Suppose that S is a smooth surface. Show that there exist (finitely many) smooth parametrizations (ϕ_j, E_j) of portions of S such that $S = \bigcup_{j=1}^N \phi_j(E_j)$.

7. Wade, 13.5.10e. Suppose that V is open and nonempty in $\mathbb{R}^2$, that u is C^2 on V, and that u is continuous on $\overline{V}$. Prove that u is harmonic on V if and only if

$$\int_{\partial E} (-u_y, u_x) \cdot \mathbf{T}\, ds = 0$$

for all two-dimensional regions $E \subseteq V$ which satisfy the hypotheses of Green's Theorem.

Check the rest of this problem in the book for a reminder as to what some of these terms mean. Also, note that I'm writing the line integral in question in vector form as opposed to in differential form as the book does, only because we didn't really talk about line integrals written in differential form until the final day of class, which I said would not be on the final.

8. Wade, 13.6.7a. Let S be an orientable surface with unit normal $\mathbf{n}$ and nonempty boundary ∂S which satisfies the hypotheses of Stokes' Theorem. Suppose that $\mathbf{F} : S \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is C^1, that ∂S is smooth, and that $\mathbf{T}$ is the unit tangent vector on ∂S induced by $\mathbf{n}$. If the angle between $T(\mathbf{x}_0)$ and $\mathbf{F}(x_0)$ is never obtuse (i.e. greater than $90°$) for any $\mathbf{x}_0 \in \partial S$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$, prove that $\mathbf{T}(\mathbf{x}_0)$ and $\mathbf{F}(\mathbf{x}_0)$ are orthogonal for all $\mathbf{x}_0 \in \partial S$.

9. Wade, 13.6.8. Suppose that E is a two-dimensional region such that if $(x, y) \in E$, then the line segments from $(0, 0)$ to $(x, 0)$ and form $(x, 0)$ to (x, y) are both subsets of E. If $\mathbf{F} : E \to \mathbb{R}^2$ is C^1, prove that the following three statements are equivalent.

(a) $\mathbf{F} = \nabla f$ on E for some $f : E \to \mathbb{R}$.

(b) $\mathbf{F} = (P, Q)$ satisfies $Q_x = P_y$. (The book call this condition being *exact*, which is a nonstandard usage of that term; most people would refer to this as saying that $\mathbf{F}$ is *closed*, which is related to the differential form approach to vector fields I outlined on the last day of class. In this setting, being *exact* is precisely the condition in part (a).)

(c) $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$ for all piecewise smooth curves $C = \partial\Omega$ oriented counterclockwise, where Ω is a two-dimensional region which satisfies the hypotheses of Green's Theorem and $\Omega \subseteq E$.

Note that I changed the original version of the problem I had to simply be the book's version instead; my version was just a little too tricky and the book's version already contains the key concepts anyway. I'll just point out, as my version suggested, that the equivalences given here hold for other types of regions more generally, such as open connected regions and so-called simply-connected ones—it's just that proving the equivalences in these settings requires more work that's not worth going through for the purposes of the final.

10. Wade, 13.6.11. Let $\mathbf{F}$ be C^1 and exact (according to the book's definition in Exercise 13.6.8) on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

(a) Suppose that C_1 and C_2 are disjoint smooth simple curves, oriented in the counterclockwise direction, and that E is a two-dimensional region whose topological boundary ∂E is the union of C_1 and C_2. If $(0, 0) \notin E$, prove that

$$\int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds.$$

(b) Suppose that E is a two-dimensional region which satisfies $(0, 0) \in E^o$. If ∂E is a smooth simple curve oriented in the counterclockwise direction and

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right),$$

compute $\int_{\partial E} \mathbf{F} \cdot \mathbf{T}\, ds$.

(c) State and prove an analogue of part a) for functions $\mathbf{F} : \mathbb{R}^3 \setminus \{\mathbf{0}\}$, three-dimensional regions, and smooth surfaces.