Math 320-1: Final Exam Solutions
Northwestern University, Fall 2013

1. For each of the statements below, decide if it is true or false. If true, prove it; if false, give a counterexample.

(a) For functions f and g on $[0,1]$,

$$\sup_{x\in[0,1]}(f(x)+g(x)) = \sup_{x\in[0,1]} f(x) + \sup_{x\in[0,1]} g(x).$$

(b) If f is uniformly continuous on the open interval $(0,1)$, then the limit $\lim_{n\to\infty} f(\frac{1}{n})$ exists.

(c) If f is integrable and nonzero on $[-2,5]$, then $\frac{1}{f}$ is integrable on $[-2,5]$.

Solution. (a) This is false. Take $f(x) = 0$ for $0 \le x < 1$ and $f(1) = 1$ and $g(x) = 0$ for $0 \le x < 1$ and $g(1) = -1$. Then $f(x) + g(x) = 0$ for all x so

$$\sup_{x\in[0,1]}(f(x)+g(x)) = 0$$

but $\sup f = 1$ and $\sup g = 0$, so

$$\sup_{x\in[0,1]} f(x) + \sup_{x\in[0,1]} g(x) = 1 + 0 = 1.$$

(b) This is true. Since f is uniformly continuous on $(0,1)$, it extends to a continuous function on $[0,1]$, which we will still call f. Then f is continuous at 0 so since $\frac{1}{n} \to 0$ we must have

$$f\left(\frac{1}{n}\right) \to f(0),$$

so $\lim_{n\to\infty} f(\frac{1}{n})$ exists.

(c) This is false. Take $f(x) = x - 5$ for all $-2 \le x < 5$ and $f(5) = 1$. Then f is integrable since it differs from the integrable function $g(x) = x - 5$ at only point point, and is nonzero since $x - 5 > 0$ for $x \ne 5$. However, f gets arbitrarily close to 0 so $\frac{1}{f}$ is unbounded on $[-2,5]$ and is thus not integrable on $[-2,5]$. □

2. Show that the function $f(x) = \frac{1}{x^2} + |x-2|$ is uniformly continuous on $[1,\infty)$.

Proof 1. Let $\epsilon > 0$ and set $\delta = \frac{\epsilon}{3}$. Then if $x, y \in [1,\infty)$ and $|x-y| < \delta$, we have

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x^2} + |x-2| - \frac{1}{y^2} - |y-2|\right| \\
&\le \left|\frac{1}{x^2} - \frac{1}{y^2}\right| + ||x-2| - |y-2|| \\
&\le \frac{|y^2 - x^2|}{|x^2y^2|} + |(x-2) - (y-2)| \\
&= |y-x|\left(\frac{y+x}{x^2y^2}\right) + |x-y| \\
&= |x-y|\left(\frac{1}{x^2y} + \frac{1}{xy^2}\right) + |x-y| \\
&\le 2|x-y| + |x-y| \\
&< 3\delta = \epsilon.
\end{aligned}$$

Thus f is uniformly continuous on $[1,\infty)$ as claimed. □

Proof 2. First, consider the function $g(x) = \frac{1}{x^2}$. Since $g'(x) = -\frac{2}{x^3}$ is bounded by 1 on $[1, \infty)$, g is uniformly continuous on $[1, \infty)$ as a consequence of the Mean Value Theorem. To be clear, for any $x, y \in [1, \infty)$ there exists $c \in (1, \infty)$ such that

$$|g(x) - g(y)| = |g'(c)||x - y| \leq |x - y|,$$

so g is Lipchitz and hence uniformly conitnuous. (We can take $\delta = \epsilon$ in the definition of uniformly continuous.)

Now, the function $h(x) = |x - 2|$ is also uniformly continuous on $[1, \infty)$. Indeed, for any $\epsilon > 0$, taking $\delta = \epsilon$ we have that $|x - y| < \delta = \epsilon$ implies

$$|h(x) - h(y)| = ||x - 2| - |y - 2|| \leq |(x - 2) - (y - 2)| = |x - y| < \epsilon,$$

so h is uniformly continuous.

Since f is the sum of two uniformly continuous functions, it itself is uniformly continuous. Indeed, let $\epsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that

$$|x - y| < \delta_1 \text{ implies } |g(x) - g(y)| < \frac{\epsilon}{2}$$

and

$$|x - y| < \delta_2 \text{ implies } |h(x) - h(x)| < \frac{\epsilon}{2}.$$

Hence for $|x - y| < \min\{\delta_1, \delta_2\}$ we have

$$|f(x) - f(y)| = |(g(x) - g(y)) + (h(x) - h(y))| \leq |g(x) - g(y)| + |h(x) - h(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so f is uniformly continuous on $[1, \infty)$ as claimed. □

3. Let f be a twice differentiable function on the interval $(0, 4)$. Suppose that $f(1) > f(2)$ and $f(3) > f(2)$. Prove that there exits a point $c \in (0, 4)$ for which $f''(c) > 0$.

Proof. By the Mean Value Theorem there exist $1 < x_1 < 2$ and $2 < x_2 < 3$ such that

$$f(2) - f(1) = f'(x_1)(2 - 1) = f'(x_1) \text{ and } f(3) - f(2) = f'(x_2)(3 - 2) = f'(x_2).$$

Since $f(1) < f(2)$ and $f(3) > f(2)$, this means that $f'(x_1) < 0 < f'(x_2)$. Note also that $x_1 < 2 < x_2$. Again by the Mean Value Theorem (only now applied to the function f') there exists $x_1 < c < x_2$ such that

$$f'(x_2) - f'(x_1) = f''(c)(x_2 - x_1).$$

Since $f'(x_1) < f'(x_2)$ and $x_1 < x_2$, this implies that $f''(c) > 0$, as desired. □

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable and that f''' is continuous. Show that $f^{(k)}(0) = 0$ for $k = 0, 1, 2$ if and only if $\lim_{x \to 0} \frac{f(x)}{x^3}$ exists.

Proof 1. Here is a proof using Taylor's theorem. For any x there exists c_x between x and 0 such that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(c_x)}{3!}x^3.$$

Then

$$\frac{f(x)}{x^3} = \frac{f(0)}{x^3} + \frac{f'(0)}{x^2} + \frac{f''(0)}{2x} + f'''(c_x).$$

Based on this, since the denominators of the first three terms go to 0 as $x \to 0$ while the numerator stays constant, the first three terms go to ∞ as $x \to 0$ unless all of these numerators are 0.

Since c_x is between x and 0, as $x \to 0$, $c_x \to 0$ as well so the last term $f'''(c_x)$ goes to $f'''(0)$ since f''' is continuous. Thus

$$\lim_{x\to 0} \frac{f(x)}{x^3} = \lim_{x\to 0} \left(\frac{f(0)}{x^3} + \frac{f'(0)}{x^2} + \frac{f''(0)}{2x} + f'''(c) \right)$$

exists if and only if the limit of the first three terms exists, which happens if and only if the numerators of the the first three terms are 0. Thus $\lim_{x\to 0} \frac{f(x)}{x^3}$ exists if and only if $f(0) = f'(0) = f''(0) = 0$ as claimed. □

Proof 2. Here is a proof using L'Hopital's rule. Note that since f is three times differentiable, f, f', and f'' are all continuous, and we are given that f''' is also continuous. Suppose that $f(0) = f'(0) = f''(0) = 0$. Since f is continuous, $f(x) \to f(0) = 0$ as $x \to 0$, so by L'Hopital's rule

$$\lim_{x\to 0} \frac{f(x)}{x^3} = \lim_{x\to 0} \frac{f'(x)}{3x^2}.$$

Since f' is continuous, $f'(x) \to f'(0) = 0$ as $x \to 0$ so again by L'Hopital's rule

$$\lim_{x\to 0} \frac{f'(x)}{3x^2} = \lim_{x\to 0} \frac{f''(x)}{6x}.$$

Since f'' is continuous, $f''(x) \to f''(0) = 0$ as $x \to 0$ so L'Hopital's rule once more implies that

$$\lim_{x\to 0} \frac{f''(x)}{6x} = \lim_{x\to 0} \frac{f'''(x)}{6}.$$

Since f''' is continuous, $f'''(x) \to f'''(0)$ as $x \to 0$ so this final limit exists, and hence $\lim_{x\to 0} \frac{f(x)}{x^3}$ exists as well.

Conversely, suppose that $\lim_{x\to 0} \frac{f(x)}{x^3}$ exists. Since the denominator goes to 0, this is only possible if the numerator goes to 0 as well. But $\lim_{x\to 0} f(x) = f(0)$ since f is continuous so we have $f(0) = 0$. By L'Hopital's rule,

$$\lim_{x\to 0} \frac{f(x)}{x^3} = \lim_{x\to 0} \frac{f'(x)}{3x^2}$$

also exists. Since the denominator goes to 0, this means that the numerator must go to 0 as well, so $f'(0) = \lim_{x\to 0} f'(x) = 0$ since f' is continuous. By L'Hopital's rule again, we have that

$$\lim_{x\to 0} \frac{f'(x)}{3x^2} = \lim_{x\to 0} \frac{f''(x)}{6x}$$

exists, so $\lim_{x\to 0} f''(x) = 0$ and this implies $f''(0) = 0$ since f'' is continuous. Thus $f(0) = f'(0) = f''(0) = 0$ as claimed.

(Note that in this converse direction we didn't need the assumption that f''' is continuous, nor even that it exists. Also, there's probably a way to combine both directions into one to avoid having to use L'Hopital's rule so many times.) □

5. Let f be the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x > 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

Show that f is integrable on $[-1, 1]$.

Proof. First, note that on $[-1,1]$ we have $-1 \le f(x) \le 1$ since $\sin\frac{1}{x}$ and $-x$ are each bounded by -1 and 1. Thus

$$|f(x) - f(y)| \le |f(x)| + |f(y)| \le 2 \text{ for all } x \in [-1,1].$$

Let $\epsilon > 0$. Since $-x$ is continuous on $[-1, -\frac{\epsilon}{12}]$, it and f are integrable there so there exists a partition P_1 of $[-1, -\frac{\epsilon}{12}]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{3}.$$

Since $\sin\frac{1}{x}$ is continuous on $[\frac{\epsilon}{12}, 1]$, it and f are integrable there so there exists a partition P_2 of $[\frac{\epsilon}{12}, 1]$ such that

$$U(f, P_2) - L(f, P_2) < \frac{\epsilon}{3}.$$

On $[-\frac{\epsilon}{12}, \frac{\epsilon}{12}]$, since $|f(x) - f(y)| \ge 2$ we have

$$\sup f - \inf f \ge 2.$$

Thus

$$(\sup f - \inf f \text{ on } [-\frac{\epsilon}{6}, \frac{\epsilon}{6}])(\text{length of } [-\frac{\epsilon}{6}, \frac{\epsilon}{6}]) \le 2\frac{\epsilon}{6} = \frac{\epsilon}{3}.$$

Hence for the partition P of $[-1,1]$ consisting of the points from P_1, $-\frac{\epsilon}{12}$, $\frac{\epsilon}{12}$, and the points from P_2, we have

$$\begin{aligned} U(f,P) - L(f,P) &= [U(f,P_1) - L(f,P_1)] + \text{ contribution from } [-\frac{\epsilon}{12}, \frac{\epsilon}{12}] + [U(f,P_2) - L(f,P_2)] \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Thus f is integrable on $[-1,1]$ as claimed. □

6. Suppose f is an integrable function on $[0,1]$ and $|f(x)| \ge \frac{1}{2}$ for all $x \in [0, \frac{1}{3}]$. Let $g(x) = f(x)^2$, which is also integrable on $[0,1]$. Prove that $\int_0^1 g(x)\,dx \ge \frac{1}{12}$.

Proof. We have

$$\int_0^1 g(x)\,dx = \int_0^{\frac{1}{3}} g(x)\,dx + \int_{\frac{1}{3}}^1 g(x)\,dx.$$

On $[0, \frac{1}{3}]$, $|f(x)| \ge \frac{1}{2}$ so $|g(x)| = |f(x)^2| \le \frac{1}{4}$. Thus

$$\int_0^{\frac{1}{3}} g(x)\,dx \ge \int_0^{\frac{1}{3}} \frac{1}{4}\,dx = \frac{1}{12}.$$

Since $g(x) \ge 0$ for all x, $\int_{\frac{1}{3}}^1 g(x)\,dx \ge 0$ so we get that

$$\int_0^1 g(x)\,dx \ge \frac{1}{12} + 0 = \frac{1}{12}$$

as claimed. □

7. Let $F(x) = \int_{x^2}^{4x^2} \sin\sqrt{t}\,dt$. Find $F'(\frac{\pi}{2})$.

Solution. Rewrite F as

$$F(x) = \int_{x^2}^{0} \sin\sqrt{t}\,dt + \int_{0}^{4x^2} \sin\sqrt{t}\,dt = -\int_{0}^{x^2} \sin\sqrt{t}\,dt + \int_{0}^{4x^2} \sin\sqrt{t}\,dt.$$

By the chain rule and the second Fundamental Theorem of Calculus, the derivative of the first term is

$$-2x \sin\sqrt{x^2} = -2x \sin|x|$$

and the derivative of the second term is

$$8x \sin\sqrt{4x^2} = 8x \sin(2|x|).$$

Thus

$$F'(x) = -2x \sin|x| + 8x \sin(2|x|) \text{ for all } x.$$

(The absolute values are there since x may be negative, so $\sqrt{x^2} = |x|$; however, I won't take any points off for not realizing this.) Hence

$$F'(\frac{\pi}{2}) = -2\frac{\pi}{2}\sin\frac{\pi}{2} + 8\frac{\pi}{2}\sin(2\frac{\pi}{2}) = -\pi.$$

□