Math 320-1: Midterm 1 Practice
Northwestern University, Fall 2013

Here are some practice problems to think about when preparing for the midterm. Of course, these can't possibly cover every type of question which could come up, but they should give a good idea of what to expect. Homework problems and examples from class would also be good to look at. Also, there are way more problems here than will actually be on the midterm, so don't get concerned about the length. Happy studying!

1. Give examples, with justification, of each of the following.
(a) A bounded subset of $\mathbb{R}$ containing only rational numbers whose supremum is π.
(b) A decreasing sequence with no convergent subsequence.
(c) A Cauchy sequence (x_n) and a bounded sequence (y_n) such that $(x_n y_n)$ is not Cauchy.

2. Suppose that x and y are real numbers such that for any $\epsilon > 0$, $|x - y| < \epsilon$. Show that $x = y$.

3. Suppose that A is a nonempty, bounded subset of $\mathbb{R}$ and let $c < 0$. Define cA to be the set

$$cA := \{cx \in \mathbb{R} \mid x \in A\}.$$

Show that $\sup(cA) = c \inf A$ using only the definition of supremum and infimum.

4. For each $n \in \mathbb{N}$ let $x_n = \frac{\cos n}{n} - \frac{1-n^2-2n}{n^2-3}$. Show that (x_n) converges to 1 using only the definition of convergence.

5. Suppose that (x_n) and (y_n) converge to x and y respectively. Give two proofs that $(x_n y_n)$ converges to xy. (Of course, this is a well-known limit law. One proof you can find in the book. For another, find where this is mentioned in the Lecture Notes.)

6. Pick positive a and b such that $a > b$. Define (a_n) and (b_n) recursively by

$$a_1 = a,\ a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_1 = b,\ b_{n+1} = \sqrt{a_n b_n}.$$

(a) Prove that (a_n) and (b_n) converge. Hint: First show that $a_n > a_{n+1} > b_{n+1} > b_n$.
(b) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

7. Show that the sequence $\left(\frac{1}{n^2}\right)$ is Cauchy using only the definition of a Cauchy sequence.

8. Show that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by showing that the sequence of partial sums converges. Hint: For any $k \in \mathbb{N}$,

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$