Math 320-1: Midterm 1 Solutions
Northwestern University, Fall 2013

1. Give an example of each of the following, or say why it is not possible to do so.
(a) A monotone sequence which does not converge.
(b) A Cauchy sequence (x_n) with a subsequence (x_{n_k}) which does not converge.
(c) A sequence (x_n) of numbers in $[-100, 200]$ with no convergent subsequence.

Solution. (a) The sequence $x_n = n$ is increasing (so monotone) and does not converge.

(b) This is not possible. Any Cauchy sequence converges, and as a result any subsequence also converges.

(c) This is not possible. Any sequence in $[-100, 200]$ is bounded, and the Bolzano-Weierstrass Theorem guarantees that such a sequence will always have a convergent subsequence. □

2. Let A be a nonempty subset of $\mathbb{R}$ such that $\inf A = 3$ and let $B = \{x^2 \mid x \in A\}$. Show that $\inf B = 9$.

Proof. First, since $\inf A = 3$, $3 \leq x$ for all $x \in A$ and thus

$$9 \leq x^2 \text{ for all } x \in A.$$

Hence 9 is a lower bound for B. This particular lower bound is clearly larger than any negative lower bound of B, so now suppose that ℓ is any nonnegative lower bound for B. Then

$$\ell \leq x^2 \text{ for all } x \in A.$$

Since $\ell \geq 0$, $\sqrt{\ell}$ is defined so

$$\sqrt{\ell} \leq x \text{ for all } x \in A.$$

(Note that any $x \in A$ is positive so $\sqrt{x^2} = x$ for such x.) Thus $\sqrt{\ell}$ is a lower bound for A and hence

$$\sqrt{\ell} \leq \inf A = 3.$$

Squaring both sides gives $\ell \leq 9$, so 9 is a lower bound of B which is larger than or equal to any lower bound of B, and thus $\inf B = 9$ as claimed. □

3. Show that the sequence $(\sqrt{n+3} - \sqrt{n})$ converges to 0. You can use whatever limit laws from the book you want.

Proof 1. We have

$$\sqrt{n+3} - \sqrt{n} = \frac{(n+3) - n}{\sqrt{n+3} + \sqrt{n}} = \frac{3}{\sqrt{n+3} + \sqrt{n}}.$$

The denominator goes to ∞ as n does, so a limit law from the book tells us that $\sqrt{n+3} - \sqrt{n} \to 0$ as claimed. □

Proof. Proof 2 We have

$$0 \leq \sqrt{n+3} - \sqrt{n} = \frac{3}{\sqrt{n+3} + \sqrt{n}} \leq \frac{3}{\sqrt{n}}.$$

The right side converges to 0 as $n \to \infty$, so the squeeze theorem gives $\sqrt{n+3} - \sqrt{n} \to 0$.

(This is fine, but if you really want to justify that the right side converges to 0 you can rewrite it as

$$\frac{3}{\sqrt{n}} = \sqrt{\frac{9}{n}}$$

and use the fact that if $x_n \to x$ then $\sqrt{x_n} \to \sqrt{x}$ as long as everything is nonnegative.) $\square$

4. Define the sequence (x_n) by setting $x_1 = 3$ and letting

$$x_{n+1} = 2 - \frac{1}{x_n} \text{ for all } n \in \mathbb{N}.$$

Show that (x_n) converges and find its limit.

Proof. We first show that (x_n) is decreasing and bounded below by 1. We have

$$x_1 = 3 > x_2 = \frac{5}{3}$$

so $x_n \geq x_{n+1}$ holds for the base case. If $x_k \geq x_{k+1}$ for some k,

$$\frac{1}{x_k} \leq \frac{1}{x_{k+1}}, \text{ so } -\frac{1}{x_k} \geq -\frac{1}{x_{k+1}}.$$

Thus

$$x_{k+1} = 2 - \frac{1}{x_k} \geq 2 - \frac{1}{x_{k+1}} = x_{k+2}.$$

We conclude by induction that $x_n \geq x_{n+1}$ for all n, so (x_n) is decreasing.

Now, $x_1 = 3 \geq 1$. If $x_k \geq 1$, then $\frac{1}{x_k} \leq 1$ so

$$x_{k+1} = 2 - \frac{1}{x_k} \geq 2 - 1 = 1.$$

By induction we have $x_n \geq 1$ for all n.

Since (x_n) is monotone and bounded, it converges, say to x. Note that since $x_n \geq 1$ for all n, $x \geq 1$ as well. The subsequence (x_{n+1}) of (x_n) then also converges to x, and since

$$x_{n+1} = 2 - \frac{1}{x_n} \to 2 - \frac{1}{x},$$

we must have

$$x = 2 - \frac{1}{x}.$$

Thus $(x-1)^2 = 0$, so $x = 1$. Hence (x_n) converges to 1. $\square$

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the condition that

$$|f(a) - f(b)| \leq 2|a-b|^2 \text{ for any } a, b \in \mathbb{R}.$$

Let (x_n) be a Cauchy sequence and define (y_n) by $y_n = f(x_n)$. Show that (y_n) is also Cauchy.

Proof. Let $\epsilon > 0$. Since (x_n) is Cauchy there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \sqrt{\frac{\epsilon}{2}} \text{ for } m, n \geq N.$$

Then if $m, n \geq N$:

$$|y_m - y_n| = |f(x_m) - f(x_n)| \leq 2|x_m - x_n|^2 < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

Thus (y_n) is Cauchy as well. $\square$