Math 320-1: Midterm 2 Practice
Northwestern University, Fall 2013

Here are some practice problems to think about when preparing for the midterm. Of course, these can't possibly cover every type of question which could come up, but they should give a good idea of what to expect. Homework problems and examples from class would also be good to look at. Also, there are way more problems here than will actually be on the midterm, so don't get concerned about the length. Happy studying!

1. Give examples, with justification, of each of the following.
(a) A bounded sequence (x_n) for which $\limsup x_n \neq \liminf x_n$.
(b) A function $f : [0,1] \to \mathbb{R}$ which is discontinuous at each $x \in [0,1]$.
(c) A continuous function which is not uniformly continuous.
(d) A function which is continuously differentiable but not twice differentiable.

2. Suppose that (x_n) is a bounded sequence. Prove that $\limsup(-x_n) = -\liminf(x_n)$. Give an example of a nonconstant sequence for which $\limsup(-x_n) = \liminf(x_n)$.

3. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = K$. Show that $L = K$. (In other words, prove that limits of functions, when they exist, are unique.)

4. Determine, with proof, the limit of $f(x) = \frac{1}{x}$ as $x \to \frac{1}{2}$.

5. A subset U of $\mathbb{R}$ is said to be *open* if for each $p \in U$ there exists $\delta > 0$ such that $(p-\delta, p+\delta) \subseteq U$. Show that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if the inverse image $f^{-1}(U)$ of any open set U is itself open. (This type of condition is taken as the definition of "continuous" in more general contexts; take a topology course to learn more.)

6. Suppose that $f : R \to \mathbb{R}$ satisfies

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

Show that if f is continuous at 0, then it is continuous on all of $\mathbb{R}$. (This is Problem 3.3.8c in Wade, which I told you to assume was true without proof on the homework.)

7. Prove that the functions $f(x) = \cos x$ and $g(x) = |x|$ are uniformly continuous on all of $\mathbb{R}$.

8. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq 2|x-y|^2 \text{ for all } x, y \in \mathbb{R}.$$

Show that f is constant. (So, the last problem on the first midterm was in some ways silly, since the only functions satisfying the given property are the constant functions, and the claim in that problem is easy for a constant function.)

9. Determine all the points at which the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x \in \mathbb{Q} \\ -(x-1)^3 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is differentiable.

10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $\lim_{x\to\infty} f'(x) = 0$. Show that $\lim_{x\to\infty}(f(x+1) - f(x)) = 0$.