Math 320-1: Midterm 2 Practice
Northwestern University, Fall 2014

1. Give an example of each of the following. Justify your answer.

(a) A function on $(1, 2)$ which is continuous but not uniformly continuous.

(b) A function which sends Cauchy sequences to Cauchy sequences but is not uniformly continuous.

(c) A function f on $\mathbb{R}$ for which $\lim_{x\to\pi} f(x)$ exists but where f is not continuous at π.

(d) A function f on $\mathbb{R}$ for which $\lim_{h\to 0} \frac{f(1+h)-f(1-h)}{2h}$ exists but $f'(1)$ does not. (The point here is that if $f'(1)$ exists, one can show that this limit does as well and equals $f'(1)$, but the existence of this limit is not enough to guarantee the existence of $f'(1)$.)

(e) A function which is twice differentiable on $\mathbb{R}$ but not three-times differentiable.

2. Show that the function f defined by $f(x) = \frac{x}{x^3-1}$ is continuous on $(1, \infty)$ by directly verifying the ϵ-δ definition of continuity. Is this function uniformly continous on $(1, \infty)$?

3. Determine, with justification, the points at which the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} (x-1)^2 & x \in \mathbb{Q} \\ (x-1)\sin\frac{1}{x-1} & x \notin \mathbb{Q} \end{cases}$$

is continuous.

4. Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if the preimage of any open subset of $\mathbb{R}$ is open. Recall that the *preimage* $f^{-1}(U)$ of a subset $U \in \mathbb{R}$ is the set of all $x \in \mathbb{R}$ which are sent to something inside U:

$$f^{-1}(U) := \{x \in \mathbb{R} \mid f(x) \in U\}.$$

(This gives a characterization of continuity solely in terms of open sets. Technically this problem is not a good problem for the purposes of reflecting exam material since I've said additional topics we spoke about, such as open sets, won't show up on exams; however, thinking about this problem might help to better understand the true meaning behind the ϵ-δ definition of continuity.)

5. Show that a function f is uniformly continuous on $\mathbb{R}$ if and only if for any sequences (x_n) and (y_n) such that $(x_n - y_n) \to 0$, we have $(f(x_n) - f(y_n)) \to 0$ as well. (This gives a characterization of uniform continuity in terms of sequences.)

6. Suppose that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $(-\infty, 0]$ and uniformly continuous on $[0, \infty)$. Show that f is uniformly continuous on $\mathbb{R}$.

(Here's the subtlety. For any $\epsilon > 0$ the fact that f is uniformly continuous on the first interval gives an appropriate δ_1 and the fact that it is uniformly continuous on the second gives an appropriate δ_2, so at first glance it might seem that to satisfy the requirement of uniform continuity on all of $\mathbb{R}$ taking $\delta = \min\{\delta_1, \delta_2\}$ would work. Certainly, if $|x - y| < \delta$ and both x, y are nonpositive, then we can apply what we know about δ_1 and if $|x - y| < \delta$ and both x, y are nonnegative, we can apply what we know about δ_2; but what happens if $|x - y| < \delta$ and one of x, y is negative and the other positive? Do you see why this matters? How might the fact that f is continuous at 0 help to overcome this?)

7. Determine where the function from Quiz 5 is differentiable, which was what that quiz was actually supposed to ask. (The TA misread the problem asked to determine where the function was continuous instead.)

8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ and $f'(a) > 0$. Show that there exists an interval around a such that $f(x) \le f(a) \le f(y)$ for any x to the left of a and y to the right of a in that interval. (Careful: don't assume that f is differentiable anywhere apart from a.)

9. Wade, 4.3.1

10. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are functions such that

$$|f(x) - f(y)| \le |g(x) - g(y)|\sqrt{|x-y|} \text{ for any } x, y \in \mathbb{R}.$$

If g is differentiable with bounded derivative on all of $\mathbb{R}$, show that f is constant. (Note: I really like this problem, since it brings together much of what we covered when speaking about differentiability.)