Math 320-3: Midterm 2 Practice Solutions
Northwestern University, Spring 2015

1. Give an example, with justification, of each of the following.

(a) A Jordan region $E \subseteq \mathbb{R}^2$ which contains a non-Jordan region $E_0 \subseteq E$.

(b) An integrable function on $[0,1] \times [0,1]$ which is discontinuous at infinitely many points.

(c) A function $f : [0,1] \times [0,1] \to \mathbb{R}$ whose iterated integrals exist but are not equal.

(d) A C^1 function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\int_{\phi([0,1]\times[0,1])} d(x,y) \neq \int_{[0,1]\times[0,1]} |\det D\phi(u,v)|\, d(u,v).$$

(e) A continuous function on the open unit disk $B_1(0,0)$ in $\mathbb{R}^2$ which is not integrable.

Solution. (a) Let $E = [0,1] \times [0,1]$ and $E_0 = E \cap \mathbb{Q}^2$. Then E is a Jordan region but E_0 is not since its boundary is all of E, which has non-zero area.

(b) The function

$$f(x,y) = \begin{cases} 1 & (x,y) = (\frac{1}{n}, \frac{1}{m}) \text{ for } n, m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

works. This was shown to be integrable on Homework 4 and is discontinuous at each point of the form $(\frac{1}{n}, \frac{1}{m})$, so is discontinuous infinitely often.

(c) The function described in the May 6th lecture notes, which is also in the book, works. Check the lecture notes or book for the details.

(d) The function $\phi(u,v) = (u \cos 4\pi v, u \sin 4\pi v)$ works. The image $\phi([0,1] \times [0,1])$ is the closed unit disk and $\det D\phi(u,v) = 4\pi u$, so the integral on the right is

$$\int_0^1 \int_0^1 4\pi u \, du \, dv = 2\pi,$$

whereas the integral on the left is the area of the unit disk, which is π. The point is that the change of variables formula does not apply here because ϕ is not one-to-one, rather ϕ essentially traces out the unit disk "twice", which is why we get twice the expected area.

(e) Any unbounded function, such as $f(x,y) = \frac{1}{1-x^2-y^2}$, works. □

2. Wade, 12.1.2c. By an interval in $\mathbb{R}^2$ we mean a set of the form

$$\{(x,c) \mid a \leq x \leq b\} \quad \text{or} \quad \{(c,y) \mid a \leq y \leq b\}$$

for some $a, b, c \in \mathbb{R}$. (So an interval is just a horizontal or vertical line segment.) Prove that every interval in $\mathbb{R}^2$ is a Jordan region.

Proof. Intervals are clearly bounded, so we need only show that they're boundaries have Jordan measure zero. However, an interval in $\mathbb{R}^2$ equals its own boundary, so we need to show that any interval has Jordan measure zero. (Note that considered as a subset of $\mathbb{R}$ instead, the boundary of an interval would only consist of two points.) We do the horizontal case, and leave the vertical case which is very similar to the reader.

Suppose that $I = \{(x,c) \mid a \leq x \leq b\}$ for some fixed $a, b, c \in \mathbb{R}$. This is contained in the rectangle $R = [a,b] \times [c, c+1]$. For a given $\epsilon > 0$, pick $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \epsilon$ and let G be the grid where we cut R up into n^2 equally-sized rectangles of area $\frac{b-a}{n^2}$. The only small rectangles R_i

which intersect the closure of I are those along the bottom of the grid, each of which have area $\frac{b-a}{n^2}$, so:

$$V(I;G) = \sum_{R_i \cap \overline{I} \neq \emptyset} |R_i| = \sum_{i=1}^{n} \frac{b-a}{n^2} = \frac{b-a}{n} < \epsilon.$$

Thus for any $\epsilon > 0$ we can find a grid G such that $V(I;G) < \epsilon$, showing that I has Jordan measure zero as claimed. Since $I = \partial I$, this shows that I is a Jordan region. □

3. Wade, 12.1.7b. Let $E \subseteq \mathbb{R}^n$. The *dilation* of E by a scalar $\alpha > 0$ is the set

$$\alpha E = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = \alpha \mathbf{z} \text{ for some } \mathbf{z} \in E\}.$$

Prove that E is a Jordan region if and only if αE is a Jordan region, in which case $\operatorname{Vol}(\alpha E) = \alpha^n \operatorname{Vol}(E)$.

Proof. Suppose that E is a Jordan region and let R be a rectangular box containing E. Then αR is a rectangular box containing αE, where we use the fact from linear algebra that scalings preserve angles and thus send rectangular boxes to rectangular boxes. For a grid G on R, let αG denote the corresponding grid on αG; to be clear, if

$$a_i = x_0 < x_1 < \cdots < x_n = b_i$$

is a partition of one of the side edges of R, then

$$\alpha a_i = \alpha x_0 < \alpha x_1 < \cdots < \alpha x_n = \alpha b_i$$

is the corresponding partition of one of the side edges of αR. Note that for a small R_i rectangular box in the grid G, we have:

$$\begin{aligned} |\alpha R_i| &= \text{ product of the lengths of the sides of } \alpha R_i \\ &= \text{ product of a bunch of terms of the form } (\alpha c_i - \alpha d_i) \\ &= \alpha^n (\text{product of terms of the form } c_i - d_i) \\ &= \alpha^n |R_i|. \end{aligned}$$

Now, let $\epsilon > 0$. Since E is Jordan measurable, ∂E has Jordan measure zero so there exists a grid G on R such that

$$V(\partial E; G) = \sum_{R_i \cap \partial E \neq \emptyset} |R_i| < \frac{\epsilon}{\alpha^n}.$$

Then:

$$V(\partial(\alpha E); \alpha G) = \sum_{\alpha R_i \cap \partial(\alpha E) \neq \emptyset} |\alpha R_i| = \sum_{R_i \cap \partial E \neq \emptyset} \alpha^n |R_i| = \alpha^n \sum_{R_i \cap \partial E \neq \emptyset} |R_i| < \alpha^n \frac{\epsilon}{\alpha^n} = \epsilon,$$

which shows that $\partial(\alpha E)$ has Jordan volume zero and hence that αE is a Jordan region. Since

$$V(\alpha E; \alpha G) = \sum_{\alpha R_i \cap \overline{\alpha E} \neq \emptyset} |\alpha R_i| = \sum_{R_i \cap \overline{E} \neq \emptyset} \alpha^n |R_i| = \alpha^n \sum_{R_i \cap \overline{E} \neq \emptyset} |R_i| = \alpha^n V(E; G),$$

we have $\inf\{V(\alpha E; \alpha G)\} = \alpha^n \inf\{V(E; G)\}$ and hence $\operatorname{Vol}(\alpha E) = \alpha^n \operatorname{Vol}(E)$ as claimed.

Conversely, if αE is Jordan measurable, then by the first part we have that

$$\frac{1}{\alpha}(\alpha E) = E$$

is Jordan measurable and that $\operatorname{Vol}(E) = \operatorname{Vol}(\frac{1}{\alpha}(\alpha E)) = \frac{1}{\alpha^n} \operatorname{Vol}(\alpha E)$ as required. □

4. Wade, 12.2.1. Compute the upper and lower sums $U(f, G_m)$ and $L(f, G_m)$ for $m \in \mathbb{N}$ where $f(x, y) = xy$ and G_m is the grid on $[0, 1] \times [0, 1]$ obtained by partitioning each side $[0, 1]$ into 2^m subintervals of equal length $\frac{1}{2^m}$. Prove that

$$\lim_{m \to \infty} U(f, G_m) - L(f, G_m) = 0.$$

Solution. To be clear, in G_m we partition each side $[0, 1]$ of the given square using

$$0 < \frac{1}{2^m} < \frac{2}{2^m} < \frac{3}{2^m} < \cdots < \frac{2^m - 1}{2^m} < 1.$$

Any small rectangle R_{ij} for $i, j = 1, \ldots, 2^m$ is thus of the form

$$R_{ij} = \left[\frac{i-1}{2^m}, \frac{i}{2^m}\right] \times \left[\frac{j-1}{2^m}, \frac{j}{2^m}\right]$$

and has area $\frac{1}{2^m}\frac{1}{2^m} = \frac{1}{4^m}$. Note that there are 4^m total small rectangles in the grid G_m.

Now, on any R_{ij} the supremum of f occurs at the upper right corner, and thus is:

$$f\left(\frac{i}{2^m}, \frac{j}{2^m}\right) = \frac{ij}{4^m}.$$

Thus the upper sum is:

$$U(f, G_m) = \sum_{i,j=1}^{2^m} \frac{ij}{4^m}\frac{1}{4^m} = \frac{1}{16^m}\left(\sum_{i=1}^{2^m} i\right)\left(\sum_{j=1}^{2^m} j\right) = \frac{1}{16^m}\left[\frac{2^m(2^m+1)}{2}\right]^2 = \frac{(2^m+1)^2}{4^{m+1}}.$$

On any R_{ij} the infimum of f occurs at the lower right corder, and so is given by:

$$f\left(\frac{i-1}{2^m}, \frac{j-1}{2^m}\right) = \frac{(i-1)(j-1)}{4^m}.$$

Thus the lower sum is (note that we are starting the indices at 0 instead to account for the $i - 1$ and $j - 1$ terms):

$$L(f, G_m) = \sum_{i,j=0}^{2^m-1} \frac{ij}{4^m}\frac{1}{4^m} = \frac{1}{16^m}\left(\sum_{i=0}^{2^m-1} i\right)\left(\sum_{j=0}^{2^m-1} j\right) = \frac{1}{16^m}\left[\frac{(2^m-1)2^m}{2}\right]^2 = \frac{(2^m-1)^2}{4^{m+1}}.$$

We compute

$$U(f, G_m) - L(f, G_m) = \frac{(2^m+1)^2}{4^{m+1}} - \frac{(2^m-1)^2}{4^{m+1}} = \frac{4 \cdot 2^m}{4^{m+1}} = \frac{1}{2^m}.$$

Thus we see that $U(f, G_m) - L(f, G_m) \to 0$ as $m \to \infty$ as required. (This shows that f is integrable, which we already knew anyway since f is continuous.) □

5. Wade, 12.2.5. If $E_0 \subseteq E$ are Jordan regions in $\mathbb{R}^n$ and $f : E \to \mathbb{R}$ is integrable on E, prove that f is integrable on E_0.

Proof. (This problem is a little tricky when it comes to getting all the details right.) Let R be a rectangular box containing E, which is thus also a rectangular box containing E_0. Let $\epsilon > 0$ and pick a grid G on R such that

$$\left| \sum_{R_i \subseteq E} M_i |R_i| - \int_E f \, dA \right| < \epsilon \quad \text{and} \quad \left| \sum_{R_i \subseteq E} m_i |R_i| - \int_E f \, dA \right| < \epsilon$$

where M_i and m_i respectively denote the supremum and infimum of f over R_i. Such a grid exists by the fact that we can approximate the integral of f to whatever degree of accuracy we want using sums involving only rectangles which are contained in the interior of E. (See Theorem 12.20 in the book.) These inequalities imply that

$$\sum_{R_i \subseteq E} (M_i - m_i)|R_i| < \left(\int_E f \, dA + \epsilon \right) - \left(\int_E f \, dA - \epsilon \right) = 2\epsilon.$$

Now, the rectangles $R_i \subseteq E$ can be separated into those fully contained in E_0 and those not fully contained in E_0, so:

$$\sum_{R_i \subseteq E_0} (M_i - m_i)|R_i| \leq \sum_{R_i \subseteq E_0} (M_i - m_i)|R_i| + \sum_{R_i \not\subseteq E_0} (M_i - m_i)|R_i| = \sum_{R_i \subseteq E} (M_i - m_i)|R_i| < 2\epsilon.$$

The same inequality will hold for any grid finer than G. Using Theorem 12.20 again only now applied to f over E_0, by making G finer if need be we get that:

$$\left| (U) \int_{E_0} f \, dA - \sum_{R_i \subseteq E_0} M_i |R_i| \right| < \epsilon \quad \text{and} \quad \left| (L) \int_{E_0} f \, dA - \sum_{R_i \subseteq E_0} m_i |R_i| \right| < \epsilon.$$

These inequalities imply that

$$(U) \int_{E_0} f \, dA - (L) \int_{E_0} f \, dA < \left(\sum_{R_i \subseteq E_0} M_i |R_i| + \epsilon \right) - \left(\sum_{R_i \subseteq E_0} M_i |R_i| - \epsilon \right) = \sum_{R_i \subseteq E_0} (M_i - m_i)|R_i| + 2\epsilon.$$

Together with the previously-derived inequality we thus have

$$(U) \int_{E_0} f \, dA - (L) \int_{E_0} f \, dA < 4\epsilon,$$

and since $\epsilon > 0$ is arbitrary this implies that

$$(U) \int_{E_0} f \, dA = (L) \int_{E_0} f \, dA,$$

showing that f is integrable over E_0. □

6. Show that the function

$$f(x,y) = \begin{cases} 1 - x - y & (x,y) = (\frac{1}{2^n}, \frac{1}{3^m}) \text{ for } n, m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

is integrable over $[0,1] \times [0,1]$.

Solution. This essentially the same argument is that on Problem 3 of Homework 4, whose solution we use as a guide. In particular, note that

$$0 \leq 1 - x - y \leq 1 \text{ for } (x, y) \text{ of the form } \left(\frac{1}{2^n}, \frac{1}{3^m}\right).$$

Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $\frac{1}{2^{N+1}} < \frac{\epsilon}{4}$ and $\frac{1}{3^{N+1}} < \frac{\epsilon}{4}$. Pick a rectangle R_{nm} of area less than $\frac{\epsilon}{2N^2}$ around each of the finitely many points $(\frac{1}{2^n}, \frac{1}{3^m})$ for $n, m = 1, \ldots, N$. If need be we can shrink each R_{nm} so that they are all disjoint.

Take a fine enough grid G on $[0, 1] \times [0, 1]$ so that the small rectangles along the left side have horizontal length $\frac{\epsilon}{4}$, all small rectangles along the bottom side have vertical height $\frac{\epsilon}{4}$, and the small rectangle containing $(\frac{1}{2^n}, \frac{1}{3^m})$ for $n, m = 1, \ldots, N$ is contained in R_{nm}. Any rectangle in our grid contains points not of the form $(\frac{1}{2^n}, \frac{1}{3^m})$, so f attains the value zero on any rectangle and hence $\inf f = 0$ on any rectangle. Thus $L(f, G) = 0$.

There are three types of rectangles contributing to $U(f, G)$: those along the left side of length $\frac{\epsilon}{4}$, those along the bottom of height $\frac{\epsilon}{4}$, and those inside some R_{nm} containing a point of the form $(\frac{1}{2^n}, \frac{1}{3^m})$ for $n, m = 1, \ldots, N$. (On any other rectangle in our grid, $\sup f = 0$ and so these give no contributions to $U(f, G)$.) On any of these rectangles we have $\sup f \leq 1$, so:

$$\begin{aligned}
\sum_{\text{rectangles } R \text{ along left}} (\sup f)(\text{area } R) &\leq \sum_{\text{rectangles along left}} (\text{area } R) = \frac{\epsilon}{4} \\
\sum_{\text{rectangles } R \text{ along bottom}} (\sup f)(\text{area } R) &\leq \sum_{\text{rectangles along bottom}} (\text{area } R) = \frac{\epsilon}{4} \\
\sum_{\text{rectangles } R \text{ inside some } R_{nm}} (\sup f)(\text{area } R_{nm}) &\leq \sum_{\text{rectangles } R \text{ inside some } R_{nm}} (\text{area } R_{nm}) \\
&< \sum_{R_{nm} \text{ for } n,m=1,\ldots,N} \frac{\epsilon}{2N^2} = \frac{\epsilon}{2},
\end{aligned}$$

where in the final equality we use the fact that there are N^2 rectangles among the R_{nm}. Thus all together we get:

$$U(f, G) < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon,$$

so $U(f, G) - L(f, G) = U(f, G) < \epsilon$ for this grid and we hence conclude that f is integrable over $[0, 1] \times [0, 1]$ as desired. □

7. Wade, 12.3.8. Let E be a nonempty Jordan region in $\mathbb{R}^2$ and let $f : E \to [0, \infty)$ be integrable on E. Prove that the volume of $\Omega = \{(x, y, z) \mid (x, y) \in E \text{ and } 0 \leq z \leq f(x, y)\}$ satisfies

$$\text{Vol}(\Omega) = \iint_E f \, dA.$$

Proof. The volume of Ω is given by:

$$\text{Vol}(\Omega) = \iiint_\Omega dV.$$

Since the constant function $\mathbf{1}$ is integrable as a function of one, two, and three variables, the iterated integrals of the above expression all exist and so Fubini's Theorem gives:

$$\iiint_\Omega dV = \iint_E \left(\int_0^{f(x,y)} dz \right) dA = \iint_E f(x, y) \, dA$$

as required. (The point of this problem was really to just understand the use of Fubini's Theorem and to set up the inner bounds on z correctly.) $\square$

8. Wade, 12.3.9ab. Let $R = [a,b] \times [c,d]$ be a two-dimensional rectangle and $f : R \to \mathbb{R}$ be bounded.

(a) Prove that

$$(L)\iint_R f\,dA \le (L)\int_a^b \left((X)\int_c^d f(x,y)\,dy\right)dx \le (U)\int_a^b \left((X)\int_c^d f(x,y)\,dy\right)dx$$

for $X = U$ or $X = L$. (So, omit the final inequality which is asked for, whose proof is similar to the first inequality asked for. This is too hard for the midterm, but good to think about nonetheless. This is similar to Lemma 12.30, only here we are not assuming that $f(x,\cdot)$ is integrable for a fixed x, which is why we're having to use upper and lower integrals for the "inner" integrals.)

(b) Prove that if f is integrable on R, then

$$\iint_R f\,dA = \int_a^b \left((L)\int_c^d f(x,y)\,dy\right)dx = \int_a^b \left((U)\int_c^d f(x,y)\,dy\right)dx.$$

(Feel free to use the additional inequality of part (a) which I omitted above. View this as a version of Fubini's Theorem which holds without assuming the single-variable function $f(x,\cdot)$ is integrable.)

Proof. (a) We mimic the proof of Fubini's Theorem we gave in class and in the lecture notes. Take a grid G on R where we partition $[a,b]$ as

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and $[c,d]$ as

$$c = y_0 < y_1 < y_2 < \cdots < y_n = b.$$

Let m_{ij} denote the infimum of f over the rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Fix $x \in [x_{i-1}, x_i]$. Then

$$m_{ij} \le f(x,y) \text{ for all } y \in [y_{j-1}, y_j]$$

so

$$(L)\int_{y_{j-1}}^{y_j} m_{ij}\,dy \le (L)\int_{y_{j-1}}^{y_j} f(x,y)\,dy$$

using the fact that the operation of taking lower integrals preserves inequalities. The lower integral on the left is that of a constant function, so in this case the lower integral is just the ordinary integral (which exists), so the left side is $m_{ij}\Delta y_j$. Summing over all intervals $[y_{j-1}, y_j]$ gives

$$\sum_j m_{ij}\Delta y_j \le \sum_j (L)\int_{y_{j-1}}^{y_j} f(x,y)\,dy = (L)\sum_j \int_{y_{j-1}}^{y_j} f(x,y)\,dy = (L)\int_c^d f(x,y)\,dy.$$

Now taking lower integrals of both sides over $[x_{i-1}, x_i]$ gives

$$(L)\int_{x_{i-1}}^{x_i} \left(\sum_j m_{ij}\Delta y_j\right)dx \le (L)\int_{x_{i-1}}^{x_i} \left((L)\int_c^d f(x,y)\,dy\right)dx.$$

Again the lower integral on the left exists as an ordinary integral since it is the lower integral of a constant function, so

$$\sum_j m_{ij}\Delta x_i \Delta y_j = \int_{x_{i-1}}^{x_i} \left(\sum_j m_{ij}\Delta y_j \right) dx \le (L) \int_{x_{i-1}}^{x_i} \left((L) \int_c^d f(x,y)\, dy \right) dx.$$

Summing over all intervals $[x_{i-1}, x_i]$ gives

$$\sum_{i,j} m_{ij}\Delta x_i \Delta y_j \le \sum_i (L) \int_{x_{i-1}}^{x_i} \left((L) \int_c^d f(x,y)\, dy \right) dx = (L) \int_a^b \left((L) \int_c^d f(x,y)\, dy \right) dx.$$

Thus the supremum of the terms on the left, which is $(L)\iint_R f\, dA$, is less than or equal to the term on the right, so

$$(L) \iint_R f\, dA \le (L) \int_a^b \left((L) \int_c^d f(x,y)\, dy \right) dx.$$

Since

$$(L) \int_a^b \left((L) \int_c^d f(x,y)\, dy \right) dx \le (L) \int_a^b \left((U) \int_c^d f(x,y)\, dy \right) dx$$

due to the integrand on the right being larger than or equal to that on the left, we have

$$(L) \iint_R f\, dA \le (L) \int_a^b \left((X) \int_c^d f(x,y)\, dy \right) dx$$

for either $X = U$ or $X = L$, which is the first desired inequality. The remaining inequalities follow from the fact that lower integrals are always smaller than or equal to upper integrals, but we'll leave it at that since this solutions is already getting long and tedious. Again, this problem is too hard for the midterm.

(b) If f is integrable on R, then

$$(L) \iint_R f\, dA = (U) \iint_R f\, dA.$$

Thus the inequality

$$(U) \int_a^b \left((X) \int_c^d f(x,y)\, dy \right) dx \le (U) \iint_R f\, dA$$

together with the inequalities in part (a) implies that all four expressions are equal, so

$$\iint_R f\, dA = (L) \int_a^b \left((X) \int_c^d f(x,y)\, dy \right) dx = (U) \int_a^b \left((X) \int_c^d f(x,y)\, dy \right) dx$$

for $X = U$ or $X = L$. Hence the iterated integrals

$$\int_a^b \left((L) \int_c^d f(x,y)\, dy \right) dx \quad \text{and} \quad \int_a^b \left((U) \int_c^d f(x,y)\, dy \right) dx$$

both exist since the lower and upper integrals for each are equal. Since

$$(L) \int_c^d f(x,y)\, dy \le (U) \int_c^d f(x,y)\, dy,$$

we have

$$\int_a^b \left((L) \int_c^d f(x,y)\,dy \right) dx \le \int_a^b \left((U) \int_c^d f(x,y)\,dy \right) dx.$$

Combined with our previous inequalities we now have

$$\iint_R f\,dA = \int_a^b \left((L) \int_c^d f(x,y)\,dy \right) dx \le \int_a^b \left((U) \int_c^d f(x,y)\,dy \right) dx = \iint_R f\,dA,$$

which finally gives

$$\iint_R f\,dA = \int_a^b \left((L) \int_c^d f(x,y)\,dy \right) dx = \int_a^b \left((U) \int_c^d f(x,y)\,dy \right) dx$$

as required. □

9. Wade, 12.4.5cd. (c) Find

$$\iint_E e^{(y-x)/(y+x)}\,dA$$

where E is the trapezoid with vertices $(1,1), (2,2), (2,0), (4,0)$.

(d) Given $\int_0^1 (1-x)f(x)\,dx = 5$, find

$$\int_0^1 \int_0^x f(x-y)\,dy\,dx.$$

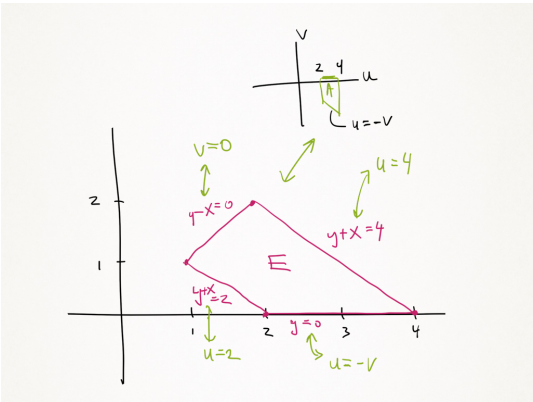
Solution. (c) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the function

$$\phi(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v) \right),$$

which is C^1, one-to-one, and has Jacobian matrix:

$$D\phi = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

at every point. (To be clear, this function sets $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$, so $u = y+x$ and $v = y-x$.) For the region A in the uv-plane bounded by the line $u = 2, u = 4, v = 0$, and $u = -v$ we have $\phi(A) = E$:



Thus by the change of variables formula we have:

$$\iint_{\phi(A)} e^{(y-x)/(y+x)}\,dA = \iint_A e^{v/u}|\det D\phi(u,v)|\,d(u,v) = \iint_A \frac{1}{2}e^{v/u}\,d(u,v).$$

Since the integrand is continuous on A, Fubini's Theorem gives

$$\iint_A \frac{1}{2}e^{v/u}\,d(u,v) = \frac{1}{2}\int_2^4\int_{-u}^0 e^{v/u}\,dv\,du = \frac{1}{2}\int_2^4 u(1-e^{-1})\,du = 3(1-e^{-1})$$

as the desired value.

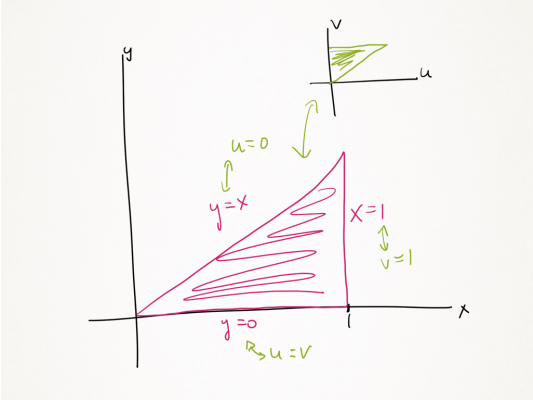
(d) Let $\phi : R^2 \to \mathbb{R}^2$ be the function

$$\phi(u,v) = (v, v-u),$$

so $x = v$ and $y = v - u$. This is C^1 and one-to-one and has Jacobian matrix

$$D\phi = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

everywhere. For the region E in the uv-plane bounded by $u = 0, u = v$, and $v = 1$ we have that $\phi(E)$ is described by the bounds on the given double integral:



Thus the change of variables formula gives:

$$\int_0^1\int_0^x f(x-y)\,dy\,dx = \iint_{\phi(E)} f(x-y)\,d(x,y) = \iint_E f(u)|\det D\phi(u,v)|\,d(u,v) = \iint_E f(u)\,d(u,v)$$

since $\det D\phi = 1$. By Fubini's Theorem the resulting integral is:

$$\iint_E f(u)\,d(u,v) = \int_0^1\int_u^1 f(u)\,dv\,du = \int_0^1 (1-u)f(u)\,du = 5$$

by the assumption that $\int_0^1 (1-x)f(x)\,dx = 5$. □

10. Wade, 12.4.7. Show that Vol is *rotation invariant* in $\mathbb{R}^2$; that is, if ϕ is a rotation on $\mathbb{R}^2$ and E is a Jordan region in $\mathbb{R}^2$, then $\text{Vol}(\phi(E)) = \text{Vol}(E)$. (See Exercise 8.2.9 to remind yourself of what rotations explicitly looks like.)

Proof. Let ϕ be the rotation given by the 2×2 matrix

$$\phi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in [0, 2\pi)$. Since ϕ is a linear transformation, it is continuously differentiable and $D\phi = \phi$, which is invertible everywhere. Thus the change of variables formula applies to give:

$$\operatorname{Vol}(\phi(E)) = \int_{\phi(E)} d\mathbf{x} = \int_E |\det D\phi(\mathbf{u})|\, d\mathbf{u} = \int_E |\det \phi|\, d\mathbf{u}.$$

By the explicit form of ϕ given above, $\det \phi = 1$ so the final integral is simply

$$\int_E d\mathbf{u} = \operatorname{Vol} E.$$

Hence $\operatorname{Vol}(\phi(E)) = \operatorname{Vol} E$ as claimed. (Thus, rotations are volume-preserving, as would be expected.) □