ANOTHER PROOF OF M. KONTSEVICH FORMALITY THEOREM FOR $\mathbb{R}^n$

DMITRY E. TAMARKIN

1. Introduction

This is a draft of paper in which we announce a plan of an alternative proof of M. Kontsevich formality theorem [7]. The basic idea is to equip Hochschild cochains of an associative algebra A with a structure of homotopy Gerstenhaber algebra and to prove the formality of this Gerstenhaber algebra. See [2]. The author was told about this idea by Boris Tsygan about a year ago.

The homological obstructions to formality vanish in the case $A = SR^n; C^\infty(R^n)$. In other words, the Gerstenhaber algebras formed by Hochschild cohomology are not deformable. The operad e_2 governing Gerstenhaber algebras is Koszul, therefore it has a canonical resolution which we denote by $\mathcal{HE}_2$. The Hochschild cochains of an associative algebra have a canonical structure of an algebra over the operad $\mathcal{B}_\infty$ (see [3] and section 2.1.5), and it suffices to construct a map $\mathcal{HE}_2 \to \mathcal{B}_\infty$. Such a map must induce the correct structure of Gerstenhaber algebra on the Hochschild cohomology and the correct Gerstenhaber bracket on Hochschild cochains. These conditions are formalized in Theorem 2.1, and in the sections 2.2.3 and 3 M. Kontsevich's theorem is deduced from Theorem 2.1.

In the section 4 we formulate Theorem 4.2 and show that it implies Theorem 2.1. The rest of the paper is devoted to the proof of this theorem.

In section 5 we construct a map $k : \mathcal{B}_\infty \to e_2$. The existence of such a map (satisfying the condition 1 of Theorem 4.2) is a direct corollary of Etingof-Kazhdan theorem on quantization of bialgebras.

In section 6 we construct the operad $\mathcal{F}$. If we new that the homology operad of $\mathcal{B}_\infty$ is e_2 we could take $\mathcal{F} = \mathcal{B}_\infty$. Let us outline the main steps of construction of $\mathcal{F}$. Suppose we have an operad $\mathcal{X}$ in the category of dg-coalgebras with counit (for example the asymmetric operad As from section 6.1.6) and a dg coalgebra with counit W. Then we can define a notion of an $\mathcal{X}$-algebra on W. It is the same as in the case of usual operads, but in addition we require that all structure maps be coalgebra morphisms. Then we consider the case when W is cofree as a graded coalgebra, that is $W = TV$, and we prove that in this case $\mathcal{X}$-algebras on W are determined by a collection of operations $TV \to V$ and that the relations between these operations are described by a certain dg-operad, which we denote by $\mathcal{O}(\mathcal{X})$.

It is clear that $\mathcal{B}_\infty$ is constructed exactly in this way from $\mathcal{X} = As$. If $\mathcal{A}$ is another operad of coalgebras with counit and $f : \mathcal{A} \to As$ a morphism, then we have the induced morphism $\mathcal{O}(f) : \mathcal{O}(\mathcal{A}) \to \mathcal{O}(As)$ and our $\mathcal{F}$ is of the form $\mathcal{O}(\mathcal{A})$. Also, we prove that if $\mathcal{A}$ is free as a dg-operad, then so is $\mathcal{O}(\mathcal{A})$.

Thus, we need an $\mathcal{A}$. We want $\mathcal{A}$ to be free as a dg operad and we want f to be a quasiisomorphism. In other words, we are going to construct a resolution of As. It is natural to take the chain operad of Stasheff associahedra. Since we need zero-ary operations we have to decompose the associahedra. This is done in section 6.3.

In the sections 6.4-6.5 we prove that the through map $\mathcal{F} \to \mathcal{B}_\infty \to e_2$ is a quasi-isomorphism. We take the filtration on $\mathcal{F}$ defined by the number of internal vertices on the trees, and consider the associated spectral sequence. This sequence is similar to the spectral sequence of manifolds with corners which are the Fulton-McPherson compactifications of configuration spaces of n points on $\mathbb{R}^2$, see [3]. The author believes that $\mathcal{F}$ is a chain operad of the operad of configuration spaces for a suitable decomposition.

Finally, in section 6.6 we construct a map $\mathcal{H}olie\{1\} \to \mathcal{F}$ that insures that this construction gives the correct Gerstenhaber bracket.

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2. Statement of the main theorem and why it implies the Formality theorem

2.1. Generalities.

2.1.1. *Conventions.* 1.We denote by k some fixed field of characteristic 0. Differentials in all complexes will have degree 1 unless the opposite is stated. As usual, for any complex $V^\bullet$ we define the shifted complex $V[k]$ so that $V[k]^i = V^{k+i}$. The grading on the dual complex V^* is defined by $V^{*k} = (V^{-k})^*$.

2. For an operad $\mathcal{O}$ we denote by $\mathcal{O}(n)$ the set of n-ary operations. For an operation $s \in \mathcal{O}(n)$, the symbol $s(x_{i_1}, \dots, x_{i_n})$ will denote the effect of the application of the permutation $(i_1, \dots, i_n)$ to s. The structure map of insertion of an n-ary operation in the i-th position of an m-ary operation $\mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m+n-1)$ will be denoted by $\circ_i^{m,n}$. Sometimes the upper indices will be omitted.

3. As usual, we denote by $\mathcal{A}ssoc$, $\mathcal{C}omm$, $\mathcal{L}ie$, e_2 the operads governing associative, commutative, Lie, and Gerstenhaber algebras respectively.

2.1.2. *Shift in operads.* For an arbitrary dg-operad $\mathcal{O}$, we are going to define an operad $\mathcal{O}\{m\}$, such that a structure of an $\mathcal{O}\{m\}$-algebra on a complex V is equivalent to a structure of an $\mathcal{O}$-algebra on $V[m]$. First, we define the operad $\mathcal{C}omm\{m\}$ as follows. We set $\mathcal{C}omm\{m\}(n) = \Lambda_n^{\otimes m}[(n-1)m]$, where Λ_n is the sign representation of S_n. Let $1_n \in \mathcal{C}omm\{m\}(n)$ be the generator. Set $\circ_1^{p,q}(1_p, 1_q) = 1_{p+q-1}$. This uniquely defines $\mathcal{C}omm\{m\}$. For an arbitrary dg-operad $\mathcal{O}$ we set $\mathcal{O}\{m\} = \mathcal{O} \otimes \mathcal{C}omm\{m\}$. That is $\mathcal{O}\{m\}(n) = \mathcal{O}(n) \otimes \mathcal{C}omm\{m\}(n)$, and the structure maps are defined as tensor products of the corresponding structure maps of $\mathcal{O}$ and $\mathcal{C}omm\{m\}$.

2.1.3. It is well known that the operads $\mathcal{A}ssoc$, $\mathcal{C}omm$, $\mathcal{L}ie$, e_2 are Koszul. Therefore, they have canonical resolutions $\mathcal{H}oassoc$, $\mathcal{H}ocomm$, $\mathcal{H}olie$, $\mathcal{H}\mathcal{E}_2$ (see [4], [3]). We follow the definitions in [3]. We denote by $\mathcal{O}^\vee$ the Koszul dual cooperad for a Koszul operad $\mathcal{O}$. By definition from [3], $\mathcal{O}^\vee$ is cogenerated by $\mathcal{O}^\vee(2) \cong \mathcal{O}(2)[1]$, and the corelation space is isomorphic to

$$\mathcal{O}(2)[1] \otimes (\mathcal{O}(2)[1] \otimes_{S_2} k(S_3))/R[2],$$

where $R[2]$ is the relation space of $\mathcal{O}$.

A homotopy $\mathcal{O}$-algebra on a complex V is the same as a differential on a cofree $\mathcal{O}^\vee$-coalgebra $S_{\mathcal{O}^\vee}V$ cogenerated by V, where S is the Schur functor. Homotopy $\mathcal{O}$-algebras are governed by an operad $\mathcal{O}'$ which is a semi-free operad generated by an S-module $\mathcal{O}^\vee[-1]$. We have $\mathcal{C}omm\{n\}^\vee = \mathcal{L}ie\{-1-n\}^*$; $\mathcal{L}ie\{n\}^\vee = \mathcal{C}omm\{-1-n\}^*$; $e_2\{n\}^\vee = e_2\{-2-n\}^*$, where $*$ denotes the linear dual cooperad.

2.1.4. *Description of* $\mathcal{H}\mathcal{E}_2$. This operad was described in [3].

A homotopy e_2-algebra on a graded vector space V is a differential on a cofree coalgebra $Q(V)$ over the cooperad $e_2^\vee$ cogenerated by V. Let $P(V) \cong S_{e_2\{-1\}^*}V$ be the cofree $e_2\{-1\}^*$-coalgebra. Then $Q(V) \cong P(V[1])[-1]$. It will be more convenient for us to use $P(V[1])$ rather then $Q(V)$. $P(V[1])$ has two co-operations $P(V[1]) \to P(V[1]) \otimes P(V[1])$ comultiplication and cobracket with their degrees $+1$ and 0 respectively, and a homotopy e_2- algebra on V is the same as a differential on $P(V[1])$.

2.1.5. *Operad* $\mathcal{B}_\infty$. This operad was described in [3]. Here we reproduce its definition in a way convenient for us. By definition, $\mathcal{B}_\infty = \mathcal{B}\{1\}$, where $\mathcal{B}$ is such an operad that a structure of a $\mathcal{B}$-algebra on a complex V is equivalent to a structure of a dg-bialgebra on TV with standard coproduct

$$\Delta e_1 \otimes e_2 \otimes \ldots \otimes e_n = \sum_{i=0}^{n} e_1 \otimes \ldots \otimes e_i \bigotimes e_{i+1} \otimes \ldots \otimes e_n, \qquad (1)$$

with the standard unit $1 \in T^0V$, with the standard counit $\epsilon : TV \to T^0V$, and with a differential $D : TV \to TV$ such that the restriction of it on $V \cong T^1V$ takes values in $V \cong T^1V$ and is equal to the differential on V as a complex. To specify a $\mathcal{B}$-algebra one has to determine the maps $m_k : T^kV \to V,\ k \geq 2$ (corestrictions of D on V) and $m_{k,l} : T^kV \otimes T^lV \to V,\ k,l \geq 1$ (corestrictions on V of the product).The operad $\mathcal{B}$ is generated by these operations. These operations should satisfy certain identities that provide the associativity of the product. The condition $D^2 = 0$ defines the differential of m_k. The condition that the product agrees with the differential defines the differential of $m_{k,l}$. The degree of $m_{k,l}$ is 0; the degree of m_k is 1. For

$\mathcal{B}_\infty(n) = \mathcal{B}(n) \otimes \Lambda_n[n-1]$, the degree of $\mu_{k,l} = m_{k,l} \otimes 1_{k+l}$ is $1-k-l$, and the degree of $\mu_k = m_k \otimes 1_k$ is $2-k$. For more details see Section 6.1

PROPOSITION 2.1. *$H^\bullet(\mathcal{B}(2))$ is generated by the class of $m_{1,1}(x_1,x_2)-m_{1,1}(x_2,x_1)$ in $H^0(\mathcal{B}(2))$ and $1/2(m_2(x_1,x_2)-m_2(x_2,x_1))$ in $H^1(\mathcal{B}(2))$. Respectively, $H^\bullet(\mathcal{B}_\infty(2))$ is generated by the class of $\mu_{1,1}(x_1,x_2)+\mu_{1,1}(x_2,x_1)$ in $H^{-1}(\mathcal{B}_\infty(2))$ and $1/2(\mu_2(x_1,x_2)+\mu_2(x_2,x_1))$ in $H^0(\mathcal{B}_\infty(2))$. These generators define an isomorphism*

$$\varkappa : H^\bullet(\mathcal{B}_\infty(2)) \cong e_2(2) \tag{2}$$

Proof. Direct computation □

2.2. Statement of the main theorem.

2.2.1. Direct check shows that the operation $m_{1,1}(x_1,x_2) - m_{1,1}(x_2,x_1) \in \mathcal{B}(2)$ satisfies the Jacoby identity and its differential is zero. Therefore, we have maps $\mathcal{L}ie \to \mathcal{B}$ and

$$\mathcal{L}ie\{1\} \to \mathcal{B}_\infty. \tag{3}$$

Also we have a map $\mathcal{L}ie\{1\} \to e_2$, corresponding to the forgetting of the commutative product. This map defines a map of the resolutions

$$\phi : \mathcal{H}olie\{1\} \to \mathcal{HE}_2. \tag{4}$$

2.2.2. *The main theorem.*

THEOREM 2.1. *There exists a morphism of operads $q : \mathcal{HE}_2 \to \mathcal{B}_\infty$, such that*

1

$$\begin{array}{ccc} \mathcal{HE}_2 & \xrightarrow{q} & \mathcal{B}_\infty \\ \uparrow \phi & & \uparrow \\ \mathcal{H}olie\{1\} & \longrightarrow & \mathcal{L}ie\{1\} \end{array} \tag{5}$$

2

$$\begin{array}{ccc} H^\bullet(\mathcal{HE}_2) & \xrightarrow{q^*} & H^\bullet(\mathcal{B}_\infty(2)) \\ \sim\downarrow & \swarrow \varkappa & \\ e_2(2) & & \end{array} \tag{6}$$

2.2.3. *Why does this theorem imply formality?* Let $A = SV$ be the symmetric algebra for a graded vector space V (the cases $A = C^\infty V$ or $A = k[[V]]$ are treated in the same way). Then $C^\bullet(A,A)$ is naturally a $\mathcal{B}_\infty$-algebra (see [3]). Hence, via q, it is also an $\mathcal{HE}_2$-algebra. The cohomology $HH^\bullet(A,A) \cong SV \otimes \wedge V$ is an e_2-algebra. Condition 2 in theorem 2.1 assures that this is the Schouten algebra. In section 3 we prove that there are no obstructions to the formality of $C^\bullet(A,A)$ as an $\mathcal{HE}_2$-algebra. Therefore $C^\bullet(A,A)$ is formal as an $\mathcal{HE}_2$-algebra.

Denote $W = C^\bullet(A,A)$, $S = HH^\bullet(A,A)$. Let $P(\bullet)$ denote the same coalgebra as in 2.1.4. Then the structures of $\mathcal{HE}_2$-algebras on W and S define differentials on $P(W[1])$ and $P(S[1])$. The formality of W implies that there exists a map of

$e_2^\vee$- coalgebras $f : P(S[1]) \to P(W[1])$, such that its restriction on $S[1]$ gives a quasiisomorphism $S[1] \to W[1]$.

Let X, Y be dg-spaces. The cocommutative coproduct on $P(X)$ defines a structure of $\mathcal{C}omm\{-1\}$-coalgebra on $P(X)$. We have a canonical inclusion of $\mathcal{C}omm\{-1\}$-coalgebras $i : S^{\geq 1}(X[1])[-1] \to P(X)$ corresponding to the 'co-forgetting' of the cocommutator on $P(X)$. One can easily prove that under any coalgebra morphism $P(X) \to P(Y)$ $i(S^{\geq 1}(X[1])[-1])$ goes to $i(S^{\geq 1}(Y[1])[-1]$ and that $i(S^{\geq 1}(X[1])[-1]$ is preserved by any coderivation of $P(X)$ (as a subspace, not pointwise). In particular, this implies that any $\mathcal{HE}_2\{-1\}$-algebra on X (which is the same as a differential on $P(X)$) defines a $\mathcal{H}olie$-algebra on X (which is a differential on $S^{\geq 1}(X[1])[-1]$). The corresponding map $\mathcal{H}olie \to \mathcal{HE}_2\{-1\}$ is induced by the map (4).

We have the following diagram of maps of $\mathcal{C}omm\{-1\}$-coalgebras.

$$\begin{array}{ccc} P(S[1]) & \longrightarrow & P(W[1]) \\ i\uparrow & & i\uparrow \\ S^{\geq 1}(S[2])[-1] & \longrightarrow & S^{\geq 1}(W[2])[-1] \end{array} \qquad (7)$$

Differentials on $S^{\geq 1}(S[2])[-1]$ and $S^{\geq 1}(W[2])[-1]$ are induced by the ones on $P(S[1])$ and $P(W[1])$. The bottom map is induced by the top one. Note that $S^{\geq 1}(S[2])[-1]$ is nothing else but the chain complex of the Schouten algebra as a Lie algebra. Condition 1 in theorem 2.1 implies that $S^{\geq 1}(W[2])[-1]$ is the chain complex of the Gerstenhaber algebra on W. The bottom map of the diagram (7) gives the quasiisomorphism of Schouten algebra and Gerstenhaber algebra since its restriction onto $S[1]$ coincides with the restriction on $S[1]$ of the top map.

3. Obstructions to the formality

3.1. **Additional gradings on $P(V[1])$.** According to the description in the section 2.1.4, an $\mathcal{HE}_2$-algebra on V is the same as a differential of $P(V[1])$. The definition of $P(V[1])$ implies that $P(V[1]) \cong S((T(V[1])/shuffles)[1])[-1]$ as a vector space. Introduce two additional gradings gr_2 and gr_3 on $P(V[1])$: by the number of cobrackets and by the number of comultiplications. The grading gr_2 is first defined to be $k-1$ on $(T^k(V[1])/shuffles)[1]$ and is extended additively on $P(V[1])$. We define $\mathrm{gr}_3|_{S^k((T(V[1])/shuffles)[1])[-1]} = k-1$. Thus, $P(V[1])$ is a 3-graded vector space. the first grading gr_1 is the grading of $P(V[1])$ as an $(e_2\{-1\}^\vee)$-coalgebra, the second grading is the number of cocommutators, and the third grading is the number of comultiplications. Comultiplication has degree $(1, 0, -1)$; cobracket has degree $(0, -1, 0)$.

3.2. **Complex of coderivations.** Coderivations of $P(V[1])$ form a Lie super-algebra. Any coderivation of $P(V[1])$ is uniquely defined by its corestriction onto $V[1]$. In other words

$$\text{Coderivations of } P(V[1]) \cong \mathrm{Hom}_k(P(V[1]), V[1]).$$

Set $\mathrm{gr}_2(V[1]) = \mathrm{gr}_3(V[1]) = 0$; $\mathrm{gr}_1(V[1]) = \mathrm{gr}(V) - 1$, where gr is the original grading on V. Then $\mathrm{Hom}_k(P(V[1]), V[1])$ becomes a trigraded space and this grading of any derivation coincides with grading of this derivation as an element of $\mathrm{Hom}_k(P(V[1]), P(V[1]))$. An e_2-algebra on V is the same as a differential of $P(V[1])$ centered at the gradings $(1, -1, 0)$ (multiplication) and $(1, 0, -1)$ (bracket). Denote the $(1, -1, 0)$-part of this differential by d_m, and the $(1, 0, -1)$-part by d_{br}. Both of these parts are also differentials (since they also determine e_2-algebras). Therefore, $((\mathrm{Hom}_k(P(V[1]), V[1])^{\mathrm{gr}_2, \mathrm{gr}_3}, [d_m, \cdot], [d_{br}, \cdot])$ is a bicomplex. Here the brackets denote the commutator in the Lie algebra of coderivations. Cohomology groups of this bicomplex in degrees less than 0 are obstructions to formality of a homotopy e_2-algebra whose cohomology algebra is V. We will prove that there is no such cohomology in the case when V is the Schouten algebra. Since this bicomplex is concentrated in negative degrees, the spectral sequence of it converges.

3.3. $E_1^{\bullet,\bullet}$. We have to compute the cohomology with respect to d_m. Let us describe the action of this differential restricted to the part of $\mathrm{Hom}_k(P(V[1]), V[1])$ with grading $\mathrm{gr}_3 = 1 - k$ which is isomorphic to

$$\mathrm{Hom}_k(S^k(T(V[1])/shuffles[1]))[-1], V[1]),$$

where all the shifts here and below are made with respect to gr_1. Let $(\mathrm{Harr}(V[1]), b)$ be the standard Harrison complex of $V[1]$ as a V-module, and V as a cocommutative coalgebra:

$$\mathrm{Harr}(V[1]) = ((T(V[1])/shuffles) \otimes V[1], b),$$

where b is induced by Hochschild differential. We define gr_1 and gr_2 on $\mathrm{Harr}(V[1])$ by setting $\mathrm{gr}_1|_{V[1]}$ to be the original grading on $V[1]$, $\mathrm{gr}_2|_V = 0$. We define gr_1 on $T(V[1])/shuffles$ as the induced grading from V and we set $\mathrm{gr}_2|_{T^k(V[1])/shuffles} = k - 1$. Then $\mathrm{Harr}(V[1])$ is a complex with respect to each of these gradings. We have

$$\mathrm{Hom}_k(S^k(T(V[1])/shuffles[1])[-1], V[1]) \cong \mathrm{Hom}_{V^{\otimes k}}(S^k(\mathrm{Harr}(V[1]))[-1], V[1]).$$

3.4. **Schouten algebra.** Let V be the Schouten algebra. Then as a commutative algebra $V = S(W)$ for a certain finite dimensional graded space W. In this case, it is well known that $\mathrm{Harr}(V[1])$ is quasiisomorphic to $W[1] \otimes V[1]$ with gr_1 induced by the gradings on V and W, and $\mathrm{gr}_2 = 0$. Therefore, our complex is quasiisomorphic to

$$\mathrm{Hom}_{V^{\otimes k}}(S^k(W[1] \otimes V[1])[-1], V[1]) = V[2] \otimes S^k((W[2])^*),$$

This cohomology has grading $(\mathrm{gr}_2 = 0, \mathrm{gr}_3 = 1 - k)$ and an element $\phi = f \otimes w_1 \otimes \ldots \otimes w_k$, $\quad f \in V; w_i \in W^*$, is presented by the following map $m_\phi : P(V[1]) \to V[1]$: m_ϕ is non zero only on $P(V[1])^{(\mathrm{gr}_2 = 0, \mathrm{gr}_3 = 1-k)} \cong S^k(V[2])[-1]$,

$$m_\phi(f_1, f_2, \ldots, f_k) = f i_{w_1} f_1 \cdot \ldots \cdot i_{w_k} f_k (-1)^{\sum \mathrm{gr}\, w_i (\mathrm{gr}\, f_1 + \ldots \mathrm{gr}\, f_{i-1})} \tag{8}$$

Thus, $E_1^{0,k} = V[2] \otimes S^{1-k}(W[2]^*)$.

3.5. $E_2^{\bullet,\bullet}$. Differential d_{br} restricted to the zeroth line $\mathrm{gr}_1 = 0$ coincides with the Lie differential with respect to the bracket. The space W is equal to $U \oplus U^*[-1]$, where $U = R^n$. Therefore $W^* = W[1]$. Hence, our $E_1^{\bullet,\bullet}$ is $SW[2] \otimes S^{\geq 1}(W[-1])$ and it is easy to see that the differential d_2 induced by d_{br} is just the de Rham differential sending W identically to $W[-1]$. Since the zeroth power in the right multiple is truncated, this complex has cohomology of grading (0,0) and no other cohomology. This cohomology means that we can deform the differential in the Schouten algebra to be the bracket with some element of it. This elements do not create an obstruction to formality.

4. First Reductions

Let us state a theorem which implies theorem 2.1

THEOREM 4.1. *There exists an operad $\mathcal{F}$, a quasiisomorphism $qis : \mathcal{F} \to e_2$, a map $r : \mathcal{F} \to \mathcal{B}_\infty$, and a map $s : \mathcal{H}olie \to \mathcal{F}$, such that the following diagrams are commutative.*

1

$$\begin{array}{ccccc} \mathcal{B}_\infty & \xleftarrow{r} & \mathcal{F} & \xrightarrow{qis} & e_2 \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{L}ie\{1\} & \longleftarrow & \mathcal{H}olie\{1\} & \xrightarrow{(4)} & \mathcal{HE}_2 \end{array} \tag{9}$$

2

$$\begin{array}{ccc} H^\bullet(\mathcal{F}(2)) & \xrightarrow{qis_*} & e_2(2) \\ \downarrow r_* & \nearrow \sim & \\ H^\bullet(\mathcal{B}_\infty(2)) & & \end{array} \tag{10}$$

4.1. **Theorem 4.1 implies theorem 2.1.** We are going to use the structure of closed model category on the category of dg k-operads, where k is a field of characteristic 0.

LEMMA 4.1. *The map (4) is a cofibration.*

Proof. It follows from the fact that $\mathcal{HE}_2$ can be obtained from $\mathcal{H}olie\{1\}$ by adding free variables and by killing the cycles. $\square$

Now, consider the diagram (9). Using RLP, we have a map $\phi : \mathcal{HE}_2 \to \mathcal{F}$, such that the composition $q = r \circ \phi$ satisfies the conditions of theorem 2.1.

4.2. **One more reduction.**

THEOREM 4.2. 1 *There exists a morphism $k : \mathcal{B}_\infty \to e_2$, such that $k_* : H^\bullet(\mathcal{B}_\infty(2)) \to H^\bullet(e_2(2))$ coincides with the map (2).*

2 *There exists an operad $\mathcal{F}$ and morphisms $s : \mathcal{H}olie\{1\} \to \mathcal{F}$ and $t : \mathcal{F} \to \mathcal{B}_\infty$, such that the diagram*

$$\begin{array}{ccc} & & e_2 \\ & \nearrow r & \uparrow k \\ \mathcal{F} & \xrightarrow{t} & \mathcal{B}_\infty \\ s\uparrow & & \uparrow (3) \\ \mathcal{H}olie\{1\} & \longrightarrow & \mathcal{L}ie\{1\} \end{array} \tag{11}$$

is commutative and

3 *the map $r = k \circ t$ is a quasiisomorphism.*

It is clear that this theorem implies theorem 4.1.

5. Proof of theorem 4.2 1

Any map $\mathcal{B}_\infty \to e_2$ defines a functor from the category of e_2-coalgebras to the category of $\mathcal{B}$-coalgebras. We will construct such a functor, and the map $\mathcal{B}_\infty \to e_2$ will be defined as a unique map producing this functor.

5.1. **Category of $\mathcal{B}_\infty$-coalgebras.** . Let V be a graded vector space. Set $\Pi TV[1] = \prod_{k=0}^{\infty} T^k V[1]$. Endow this space with the p-adic topology in which the base of open neighborhoods of 0 is formed by the subspaces $\prod_{k=l}^{\infty} T^k V[1]$. The space $\Pi TV[1]$ has a natural structure of a topological algebra with unit. Indeed, the multiplication is extended by continuity from the one on $TV[1]$, and the unit is $1 \in T^0 V[1]$. Define $\epsilon : \Pi TV[1] \to k$ as the projection on $T^0 V[1]$.

PROPOSITION 5.1. *The category of $\mathcal{B}_\infty$-coalgebras is equivalent to the category whose objects are topological dg-bialgebras isomorphic to $\Pi TV[1]$ as algebras with unit and having ϵ as a counit. Morphisms between $\Pi TV[1]$ and $\Pi TW[1]$ are maps $V \to W$ which induce a morphism of topological dg bialgebras with unit and counit.*

5.2. **Construction of the Functor.**

5.2.1. *Etingof-Kazhdan theorem.* We are going to use a result from [6]. Let us reformulate it. Let $\mathfrak{a}$ be a Lie bialgebra with commutator $[,]$ and cocommutator δ. Let $U(\mathfrak{a})$ be the enveloping algebra of $\mathfrak{a}$ as a Lie algebra.

THEOREM 5.1. *(Etingof-Kazhdan) There exist $k[[h]]$-linear maps $m : U(\mathfrak{a}) \otimes U(\mathfrak{a})[[h]] \to U(\mathfrak{a})[[h]]$, $\Delta : U(\mathfrak{a})[[h]] \to U(\mathfrak{a}) \otimes U(\mathfrak{a})[[h]]$ such that*

1 $(U(\mathfrak{a})[[h]], m, \Delta)$ *with the standard unit* $1 \in U(\mathfrak{a})[[h]]$*and counit* $\epsilon : U(\mathfrak{a})[[h]] \to k[[h]]$ *is a bialgebra with unit and counit over* $k[[h]]$.
2 *Via the PBW identification* $U(\mathfrak{a}) \sim S(\mathfrak{a})$, *the components of* m $m^r_{p,q,l} : S^p\mathfrak{a} \otimes S^q\mathfrak{a} \to S^r(\mathfrak{a})h^l$ *and of* Δ $\Delta^{p,q}_{r,l} : S^r(\mathfrak{a}) \to S^p(\mathfrak{a}) \otimes S^q(\mathfrak{b})h^l$ *are obtained from the operations* $[,]$ *and* $h\delta$ *via the acyclic calculus over* $\mathbb{Q}$ *(without using* h*).*
3 *The multiplication coincides with the usual multiplication on* $U(\mathfrak{a})[[h]]$ *up to* $O(h)$.
4 Δ *is the standard coproduct on* $U(\mathfrak{a})[[h]]$ *up to* $O(h)$
5 *On* $\mathfrak{a} \subset U(\mathfrak{a})$ *we have* $\Delta\mathfrak{a} - \Delta^{op}\mathfrak{a} = h\delta a + O(h^2)$

One sees that this theorem is also applicable to Lie dg bialgebras. In this case we obtain a structure of a dg-bialgebra with unit and counit on $U(\mathfrak{a})[[h]]$, and the differential on $U(\mathfrak{a})[[h]]$ is induced from the one on $\mathfrak{a}$.

5.2.2. *A dg Lie bialgebra* $\mathcal{F}reeLieV[1]$*.* We construct a dg Lie bialgebra to which we will apply the Etingof-Kazhdan theorem.

Let V be a dg e_2-coalgebra. This means that we have a cocommutative coproduct $\Delta : V \to V \otimes V$, a cocommutator $\delta : V[1] \to V[1] \otimes V[1]$ and a differential $d : V \to V$. In particular, $V[1]$ is a dg Lie coalgebra. Let $\mathcal{F}reeLieV[1]$ be the free Lie algebra generated by $V[1]$. Define a cocommutator $\bar{\delta}$ on it as a unique map $\bar{\delta} : \mathcal{F}reeLieV[1] \to \mathcal{F}reeLieV[1] \otimes \mathcal{F}reeLieV[1]$ such that

1 $\bar{\delta}([x, y]) = [\bar{\delta}x, y] + [x, \bar{\delta}y]$;
2 $\bar{\delta}|_{V[1]} = \delta$.

One can check that $\bar{\delta}$ turns $\mathcal{F}reeLieV[1]$ into a Lie bialgebra. This bialgebra was introduced in [1]. The coproduct Δ and the differential d define a bar-differential b on the Harrison complex of a dg cocommutative coalgebra V. This complex, as a vector space, is isomorphic to $\mathcal{F}reeLieV[1]$. One can check that b is compatible with the structure of a Lie bialgebra on $\mathcal{F}reeLieV[1]$. Thus, $\mathcal{F}reeLieV[1]$ has a natural structure of a dg Lie bialgebra.

5.2.3. *Application of the Etingof-Kazhdan Theorem.* The Etingof-Kazhdan theorem gives a structure of a dg bialgebra on $U(\mathcal{F}reeLieV[1])[[h]] \cong TV[1][[h]]$ with respect to the grading gr induced form the one on V and such that gr $h = 0$. The differential $\bar{b}$ is induced from the bar-differential b on $\mathcal{F}reeLieV[1]$ and is nothing else but the bar differential of V as an associative dg coalgebra.

Note that $\mathcal{F}reeLieV[1]$ has another grading $|\ |$ such that $|[e_1, [e_2, \ldots, e_k] \ldots]| = k$. In this grading $|[,]| = 0$ and $|\delta| = 1$. The differential can be split into two parts $b = b_0 + b_1$ with gradings 0 and 1 respectively (b_0 corresponds to the differential on V and b_1 corresponds to the commutative coproduct Δ). This grading naturally extends to $U(\mathcal{F}reeLieV[1]) \cong TV[1]$ in such a way that $|e_1 \otimes \ldots \otimes e_k| = k$. Set $|h| = -1$. Then the gradings of the product and the coproduct on $TV[1][[h]]$ are equal to 0. This allows one to set $h = 1$ and to extend the product and the coproduct on $\Pi TV[1]$ by continuity. The bar-differential $\bar{b}$ can be also extended to $\Pi TV[1]$ by continuity. Therefore, we have obtained a structure of a topological dg-bialgebra on $\Pi TV[1]$. The only thing that prevents it from being a $\mathcal{B}_\infty$-algebra is the fact that the product m on $\Pi TV[1]$ is non-standard one. The Etingof-Kazhdan theorem implies the following:

1 m has a unit $1 \in \Pi TV[1]$;
2 the counit map $\epsilon : \Pi TV[1] \to k$ is a morphism of algebras;
3 for $x \in T^k V[1]$, $y \in T^l V[1]$, we have $m(x, y) = x \otimes y + z$, where $z \in T^{>k+l} V[1] \subset \Pi TV[1]$.

The last statement follows from theorem 5.1, 3.

Any product m satisfying these conditions is isomorphic to the standard one. Denote $x * y = m(x, y)$. The isomorphism ϕ is given by the formula $\phi(e_1 \otimes \ldots \otimes e_k) = e_1 * \ldots * e_k,\ k \geq 1,\ e_i \in V[1]$; $\phi(1) = 1$. Since $\phi(e_1 \otimes \ldots \otimes e_k) \in \Pi T^{\geq k} V[1]$, ϕ is well defined on $\Pi TV[1]$. Also, for an x such that $|x| = k$, we have $\phi(x) = x + y$, $y \in \Pi T^{>k} V[1]$. Therefore, ϕ^{-1} is a continuous map $\Pi TV[1] \to \Pi TV[1]$. We can conjugate all structure maps by ϕ and we obtain a dg-bialgebra $(\Pi TV[1], \phi_* m, \phi_* \Delta, \phi_* d)$ which gives a structure of a $\mathcal{B}_\infty$-algebra on V. Thus, we have constructed a functor providing a map $\mathcal{B}_\infty \to e_2$. The quasi-classical limits from theorem 5.1 4,5 allow one to easily check the last condition of theorem 4.2 1.

6. Proof of Theorem 4.2,2: construction of $\mathcal{F}$

We will start with a description of a functor from a certain sub-category of the category of operads of dg coalgebras with counit to the category of dg operads such that both $\mathcal{B}_\infty$ and $\mathcal{F}$ are in the image of this functor.

6.1. Description of the functor.

6.1.1. *Operads of coalgebras and algebras over them.* Let $\mathcal{X}$ be an asymmetric operad in the symmetric monoidal category of associative coalgebras with counit such that $\mathcal{X}(0) = \mathcal{X}(1) = k$. Let $\mathcal{C}$ be the category of such operads whose morphisms act identically on the spaces of unary and 0-ary operations. Let W be a dg-coalgebra with counit. Let $e \in W$ be a marked element.

DEFINITION 6.1. *An $\mathcal{X}$-algebra with unit e on W is a collection of maps $\phi_n : \mathcal{X}(n) \otimes W^{\otimes n} \to W$ such that*

1 *Each ϕ_n is a morphism of coalgebras with counit.*
2 *Collection of ϕ_n is a representation of $\mathcal{X}$ as a dg operad on a dg space W.*
3 *$\phi_0(1, 1) = e$; $\phi_1(1, x) = x$*

Let V be a complex. Let TV be a coalgebra with coproduct, counit, unit, and a differential described in section 2.1.5. Note that the differential is not defined uniquely.

DEFINITION 6.2. *An $\mathcal{XB}$-algebra on V is an $\mathcal{X}$-algebra with unit on TV.*

6.1.2. *Description of $\mathcal{XB}$-algebras.* To define an $\mathcal{XB}$-algebra on V one has to specify a differential $D : TV \to TV$ satisfying the conditions from section 2.1.5 and maps $\phi_n : \mathcal{X}(n) \otimes TV^{\otimes n} \to TV$ satisfying 6.1. Let us decompose D and ϕ_n into their components $D_k^l : T^k V \to T^l V$ and $\phi^r_{k_1,\dots,k_n} : T^{k_1}V \otimes \ldots \otimes T^{k_n}V \to T^r V$.

LEMMA 6.3. *We have*

1 $\phi^r_{k_1\dots k_n} = 0$ *for* $r > k_1 + \ldots + k_n$*;*
2 $D_k^l = 0$ *for* $l > k$*;* D_k^k *is the differential induced by the differential* d *on* V*.*

Proof. 1. We will use double induction with respect to $k_1 + \ldots + k_n$ and n.

a) $n = 0, 1$. In this cases the statement follows from the Definition 6.1, 3;

b) Suppose that the statement has been proven for all $n \le N$, $N \ge 2$, and any sets of $k_1 \dots k_n$ and for $n = N$ under condition that $k_1 + \ldots + k_n < M$. Let us prove the statement for $n = N$, $k_1 + \ldots + k_n = M$.

i) $M = 0, 1$. In this case one of the numbers $k_1 \dots k_n$ must be equal to 0. Let $k_i = 0$. Let $x_i \in T^{k_i}V$ and $a \in \mathcal{X}(n)$. Then

$$\phi^r_{k_1\dots k_n}(a, x_1, \dots, x_n) = x_i \phi^r_{k_1\dots\hat{k}_i\dots k_n}(\circ_i^{n,0}(a, 1), x_1, \dots \hat{x}_i \dots, x_n) = 0$$

if $r > k_1 + \ldots + k_n$ by the induction assumption.

ii) $M > 1$. Let $x_i \in T^{k_i}V$ and $a \in \mathcal{X}(n)$. Let $\Delta x_i = \sum \Delta^{p, k_i - p} x_i$, where $\Delta^{p,k_i-p} x_i \in T^p V \otimes T^{k_i - p} V$. Then

$$\begin{aligned} &\Delta \phi_n(a, x_1, \dots, x_n) - 1 \otimes \phi_n(a, x_1, \dots, x_n) - \phi_n(a, x_1, \dots, x_n) \otimes 1 \\ &= \sum \phi^i_{p_1\dots p_n} \otimes \phi^j_{k_1-p_1\dots k_n-p_n}(\Delta a, \Delta^{p_1,k_1-p_1}x_1, \dots, \Delta^{p_n,k_n-p_n}x_n), \quad (12) \end{aligned}$$

where the sum is taken over all $i, j, p_1, \dots, p_n$ such that $0 \le p_i \le k_i$, not all p_i are equal to 0, and not all p_i are equal to k_i. By the induction assumption the right hand side belongs to $\bigoplus_{r+s \le k_1+\ldots+k_n} T^r V \otimes T^s V$. Therefore, $\phi_n(a, x_1, \dots, x_n) \in T^{\le k_1+\ldots+k_n}V$. This proves the first statement of Lemma. The second statement can be proved similarly. ∎

PROPOSITION 6.4. *A structure of an $\mathcal{XB}$-algebra on a complex (V, d) is the same as a collection of maps $\phi^r_{k_1\dots k_n} : \mathcal{X}(n) \otimes T^{k_1}V \otimes \ldots \otimes T^{k_n}V \to T^r V$, $r \le k_1 + \ldots + k_n$, $k_i \ge 0$, $n \ge 0$ of degree 0 and $D_l^m : T^l V \to T^m V$, $l > m > 0$ of degree 1 satisfying the following identities. Let $x_i \in T^{k_i}V$ and $a \in \mathcal{X}(n)$.*

1 *For any i and r,*

$$\phi^r_{k_1\dots k_n}(a,x_1,\dots,x_n) = \sum_{p_1\dots p_n} \phi^i_{p_1\dots p_n}\otimes \phi^{r-i}_{k_1-p_1\dots k_n-p_n}(\Delta a, \Delta^{p_1,k_1-p_1}x_1,\dots,\Delta^{k_n,p_n-k_n}x_n). \tag{13}$$

$$D_l^{p+l-r}(x_l) = (D^p_r\otimes \mathrm{Id})(\Delta^{r,l-r}x) \pm (\mathrm{Id}\otimes D^{l-r}_{l-p})(\Delta^{p,l-p}x). \tag{14}$$

These identities express the fact that ϕ_n are coalgebra morphisms and that D is a derivation of the coalgebra TV.

2 *Let $b\in\mathcal{X}(i)$ and $c\in\mathcal{X}(n-i)$. Then*

$$\phi^r_{k_1\dots k_n}(\circ_j^{n-i,i}(c,b),x_1,\dots,x_n) = \sum_{s\le k_j+\dots+k_{j+i-1}} \phi^{n-i}_{k_1,\dots,k_{j-1},s,k_{j+i},\dots,k_n}(c,x_1,\dots,x_{j-1},\phi^s_{k_j\dots k_{j+i-1}}(b,x_j,\dots,x_{j+i-1}),x_{j+i},\dots,x_n) \tag{15}$$

3 *The map ϕ_0 maps $\mathcal{X}(0)\otimes T^0V$ identically to T^0V. The maps ϕ^i_j are identical if $i=j$ and equal to zero otherwise. We have $\phi^0_{k_1\dots k_n}(a,x_1,\dots,x_n)=0$ if not all of k_i are zeros. Otherwise, $\phi^0_{0\dots 0}(a,x_1,\dots,x_n)=x_1\cdot\dots\cdot x_n\varepsilon(a)$.*

4 $dD^k_l x + D^k_l dx + \sum_{l<r<k} D^k_r D^r_l x = 0$

5

$$d\phi^r_{k_1\dots k_n}(a,x_1,\dots,x_n) - \phi^r_{k_1\dots k_n}(da,x_1,\dots,x_n) - \sum (-1)^{|a|+|x_1|+\dots+|x_{l-1}|}\phi^r_{k_1\dots k_n}(a,x_1,\dots,dx_l,\dots,x_n)$$
$$= -\sum_s D^r_s\phi^s_{k_1\dots k_n}(a,x_1,\dots,x_n) + \sum (-1)^{|a|+|x_1|+\dots+|x_{l-1}|}\phi^r_{k_1\dots s\dots k_n}(a,x_1,\dots,D^s_l x_l,\dots,x_n) \tag{16}$$

Proof. Clear □

PROPOSITION 6.5. . *$\mathcal{X}\mathcal{B}$-algebras are governed by a certain dg PROP $P(\mathcal{X})$ generated by*

1 *the operations $\phi(x)^r_{k_1\dots k_n}$, $r\le k_1+\dots+k_n$, $k_i\ge 0$, $n\ge 0$, where $x\in\mathcal{X}(n)$ is a homogeneous element. Degree of such an operation is equal to $|x|$.*
2 *D^m_l, $l>m>0$ of degree 1.*

The relations are expressed in Proposition 6.4 1-3. The differential is defined in Proposition 6.4 4,5.

Proof. The only thing that has to be checked here is that the differential respects the relations in $P(\mathcal{X})$ and that its square is equal to zero modulo the relations in $P(\mathcal{X})$ which is obvious. □

6.1.3. *$P(\mathcal{X})$ is generated by an operad.*

LEMMA 6.6. *In Proposition 6.4 2 it suffices to set $r = 1$.*

Proof. Conditions 1,3 in Proposition 6.4 imply that after summation of (13) over r, we will get on both sides morphisms of coalgebras with counit $\mathcal{X}(n) \otimes TV^{\otimes n} \to TV$. Any such a morphism is uniquely defined by its corestriction on the cogenerators $V \in TV$. □

PROPOSITION 6.7. *The PROP $P(\mathcal{X})$ is freely generated by an operad $\mathcal{O}(\mathcal{X})$ formed by its $(n, 1)$-ary operations. (The operad $\mathcal{O}(\mathcal{X})$ does not have to be free).*

Proof. The operad $\mathcal{O}(\mathcal{X})$ is generated by $\phi(x)^1_{k_1\ldots k_n}$ with $k_1 + \ldots + k_n \geq 1$ and D^1_l, $l > 1$. Conditions 1,3 in Proposition 6.4 allow one to express the other operations in the PROP in terms of these ones. They impose no restrictions on the operad $\mathcal{O}(\mathcal{X})$. Due to the Lemma 6.6, condition 2 in Proposition 6.4 can be expressed in terms of $\mathcal{O}(\mathcal{X})$. □

6.1.4. *The case when $\mathcal{X}$ is semi-free as a dg operad.*

PROPOSITION 6.8. *Assume $\mathcal{X}(n)$, $n \geq 1$ is semi-free as a dg operad and is freely generated by subspaces $\mathcal{V}(n) \subset \mathcal{X}(n)$. Then the PROP $P(\mathcal{X})$ and the operad $\mathcal{O}(\mathcal{X})$ are also semi-free and are freely generated by $\phi(v)^1_{k_1\ldots k_n}$, $v \in \mathcal{V}(n)$, $k_1, \ldots, k_n \geq 1$, $n \geq 2$ and D^1_k, $k > 1$.*

Proof. All relations in $\mathcal{O}(\mathcal{X})$ are expressed in Proposition 6.4. Now we can uniquely restore all operations in $\o(\mathcal{X})$ by induction on the number of arguments of an operation in $\o(\mathcal{X})$. □

6.1.5. *Functor $\mathcal{O}()$.* Obviously, $\mathcal{O}()$ is a functor form the category $\mathcal{C}$ described in 6.1.1 to the category of dg operads.

6.1.6. *An asymmetric operad As and the operad $\mathcal{B}$.* An asymmetric operad As governing associative algebras can be enriched in the following way. First we can turn it into an operad with 0-ary operations by setting $As(0) = k$ and by setting the structure maps $\circ_r^{p,q} : As(p) \otimes As(q) \to As(p + q - 1)$, $p, q, p + q - 1 \geq 0$, $1 \leq r \leq p$ to be $\circ_r^{p,q}(1 \otimes 1) = 1$. Define coalgebra structure on each of $As(k)$ by $\Delta 1 = 1 \otimes 1$. This turns As into an operad with 0-ary operations of coalgebras with counit. It is clear that $\mathcal{B} = \mathcal{O}(As)$.

6.2. **Operad $\mathcal{A}$.** We will construct an operad of dg coalgebras with counit $\mathcal{A}$ with the following properties:

1 $\mathcal{A}$ is free as an operad of graded vector spaces;
2 the counit map $\varepsilon : \mathcal{A} \to As$ is a quasiisomorphism of operads of dg coalgebras with counit.

Then we will define $\mathcal{F}$ as $\mathcal{O}(\mathcal{A})\{1\}$. The map $\mathcal{O}(\varepsilon)\{1\}$ will give us a map $\mathcal{F} \to \mathcal{B}_\infty$. We will construct $\mathcal{A}$ as an operad of chain complexes of Stasheff associahedra with respect to a suitable polyhedral decomposition.

6.3. **Stasheff topological operad as an operad with 0-ary operations.** Let $\mathcal{K}$ be the Stasheff topological asymmetric operad of associahedra. Set $\mathcal{K}(1) = \mathcal{K}(0) = \mathrm{pt}$ and define on $\mathcal{K}$ a structure of an asymmetric operad with 0-ary operations. For this we need a polyhedral decomposition of $\mathcal{K}$. The structure maps will be defined to be piecewise linear with respect to this decomposition. Here and below 'piecewise linear map with respect to decomposition' means 'piecewise-linear map, mapping an element of the decomposition *onto* an element of a decomposition'.

6.3.1. *Polyhedral decomposition of $\mathcal{K}$.* Let us construct it by induction.
1) $n \le 2$. $\mathcal{K}(n) = \mathrm{pt}$ and is decomposed trivially.
2) Suppose that all $\mathcal{K}(i)$, $i \le n$ have been decomposed so that all operadic maps $\circ_l^{p,q} : \mathcal{K}(p) \times \mathcal{K}(q) \to \mathcal{K}(p+q-1)$, $1 \le p, q, p+q-1 \le n$ are piecewise linear with respect to this decomposition. Then
i) decompose the boundary $\partial\mathcal{K}(n+1)$ which is the union of $n-2$-dimensional faces of the form $\mathcal{K}(i) \times \mathcal{K}(j)$, $i+j = n+2$, $i, j \ge 2$. Decompose each such a face as a product of already decomposed spaces $\mathcal{K}(i)$ and $\mathcal{K}(j)$. Let us check that these decompositions agree on intersections of the faces. Any such an intersection is a triple product $\mathcal{K}(p) \times \mathcal{K}(q) \times \mathcal{K}(r)$, $p+q+r = n+3$, $p, q, r \ge 2$. The inclusions of these intersections into the faces look like

$$\begin{array}{ccc} (\mathcal{K}(p) \times \mathcal{K}(q)) \times \mathcal{K}(r) & \overset{\circ^{p,q}\times \mathrm{Id}}{\hookrightarrow} & \mathcal{K}(p+q-1) \times \mathcal{K}(r) \\ \downarrow\sim & & \\ \mathcal{K}(p) \times (\mathcal{K}(q) \times \mathcal{K}(r)) & \overset{\mathrm{Id}\times\circ^{q,r}}{\hookrightarrow} & \mathcal{K}(p) \times \mathcal{K}(q+r-1) \end{array} \tag{17}$$

By the induction assumption, each of these inclusions is a piecewise linear map, therefore, the decompositions do agree.

ii) Let O_{n+1} be the center of $\mathcal{K}(n+1)$. Decompose $\mathcal{K}(n+1)$ as the union of cones with vertex O_{n+1} and bases the polyhedra of the decomposition of $\partial\mathcal{K}(n+1)$. It is clear that the operadic maps $\circ_l^{p,q} : \mathcal{K}(p) \times \mathcal{K}(q) \to \mathcal{K}(p+q-1)$, $1 \le p, q, p+q-1 \le n+1$ are piecewise linear with respect to such a decomposition. Therefore the induction assumption is true.

6.3.2. *0-ary operations.* We need to define maps $\circ_j^{i,0} : \mathcal{K}(i) \times \mathcal{K}(0) \to \mathcal{K}(i-1)$, $1 \le j \le i$. Again, we will do it by induction.

1) Define $\circ_\bullet^{\bullet,0}$ on $\mathcal{K}(2)$ and $\mathcal{K}(1)$ as identical maps $\mathrm{pt} \to \mathrm{pt}$.

2) Suppose that $\circ_j^{i,0}$ have been defined for $i<n$ so that all $\circ_j^{p,q}:\mathcal{K}(p)\times\mathcal{K}(q)\to\mathcal{K}(p+q-1)$, $0\le p,q,p+q-1<n$, satisfy the operadic identities whenever all terms in them are $\mathcal{K}(i)$, $0\le i<n$. Also suppose that each $\circ_j^{i,0}:\mathcal{K}(i)\times\mathcal{K}(0)\to\mathcal{K}(i-1)$ maps the center $O_i\times\mathrm{pt}\in\mathcal{K}(i)\times\mathcal{K}(0)$ to the center O_{i-1} of $\mathcal{K}(i-1)$.

i) Construct $\circ_p^{N,0}$ on the boundary of $\mathcal{K}(n)$ as follows. Each face on the boundary is $F=\circ_j^{l,m}(\mathcal{K}(l)\times\mathcal{K}(m))$. Define $\circ_p^{n,0}$ on F to be $\mathcal{K}(l)\times(\mathcal{K}(m)\times\mathcal{K}(0))\xrightarrow{\circ_{p-j+1}^{m,0}\times\mathrm{Id}}\mathcal{K}(l)\times\mathcal{K}(m-1)\xrightarrow{\circ_j^{l-1,m}}\mathcal{K}(l+m-2)$ if $1\le p-j+1\le m$; otherwise define $\circ_p^{n,0}$ as $(\mathcal{K}(l)\times\mathcal{K}(0))\times\mathcal{K}(m)\xrightarrow{\circ_\sigma^{l,0}\times\mathrm{Id}}\mathcal{K}(l-1)\times\mathcal{K}(m)\xrightarrow{\circ_{j'}^{l-1,m}}\mathcal{K}(l+m-2)$, where $\sigma=p$, $j'=j-1$ if $p<j$; $\sigma=p-m+1$, $j'=j$ if $p>j+m-1$. The induction assumption implies that thus constructed maps agree on the intersections of the faces and define a map $\partial\mathcal{K}(n)\to\mathcal{K}(n-1)$.

ii) continue $\circ_j^{n,0}$ from $\partial\mathcal{K}(n)$ to $\mathcal{K}(n)$ by sending the center $O_n\in\mathcal{K}(n)$ to the center $O_{n-1}\in\mathcal{K}(n-1)$ and by linear continuation on each element of the decomposition of $\mathcal{K}(n)$. Let us check that $\circ_j^{n,0}$ satisfy the induction assumption. We need to check the operadic identities. We have two cases.

a)

$$\begin{array}{ccccc}(\mathcal{K}(i)\times\mathcal{K}(0))\times\mathcal{K}(j) & \longrightarrow & \mathcal{K}(i-1)\times\mathcal{K}(j) & \to & \mathcal{K}(i+j-2)\\ \downarrow & & & \nearrow & \\ (\mathcal{K}(i)\times\mathcal{K}(j))\times\mathcal{K}(0) & \longrightarrow & \mathcal{K}(i+j-1)\times\mathcal{K}(0) & & \end{array}\qquad(18)$$

b)

$$\begin{array}{ccccc}\mathcal{K}(i)\times(\mathcal{K}(j)\times\mathcal{K}(0)) & \to & \mathcal{K}(i)\times\mathcal{K}(j-1) & \to & \mathcal{K}(i+j-2)\\ \downarrow & & & \nearrow & \\ (\mathcal{K}(i)\times\mathcal{K}(j))\times\mathcal{K}(0) & \to & \mathcal{K}(i+j-1)\times\mathcal{K}(0) & & \end{array}\qquad(19)$$

We need to check that these diagrams are commutative. If $i,j<n$, then the commutativity follows immediately. The only cases that are left are

1) $i=n$, $j=0$. We have two maps $\mathcal{K}(n)\times\mathcal{K}(0)\times\mathcal{K}(0)\to\mathcal{K}(n-2)$. The induction assumption implies that they coincide on $\partial\mathcal{K}(n)$ and that they map O_n to O_{n-2}. Since both maps are piecewise linear, they coincide. 2) $i=n$, $j=1$. This is obvious.

6.3.3. *Coalgebra structure on $C_\bullet(\mathcal{K})$.* Let $C_\bullet(\mathcal{K}(n))$ be the chain complex of $\mathcal{K}(n)$ with respect to the polyhedral decomposition. Since the operadic maps are piecewise linear, these complexes form a dg-operad.We are going to endow this operad with a structure of an operad of coalgebras. To do it we need some preparation. In sections 6.3.4-6.3.7 all differentials have degree -1.

6.3.4. Let $dgcoalg_1$ be the category of dg coalgebras with counit. Then all small direct limits exist in it and commute with the forgetful functor $dgcoalg_1\to complexes$

6.3.5. *Cone over coalgebra.* Define a coalgebra $C_\bullet(I)$ such that $C_0(I)$ is spanned by 2 elements a and b; $C_1(I)$ is spanned by an element c, and $da = db = 0$; $dc = a - b$; $\Delta a = a \otimes a$; $\Delta b = b \otimes b$; $\Delta c = c \otimes a + b \otimes c$; $\varepsilon(a) = \varepsilon(b) = 1$; $\varepsilon(c) = 0$.For a dg coalgebra with counit A set Cyl $A = C_\bullet(I) \otimes A$ and Con A to be the limit of the diagram

$$\begin{array}{ccc} \text{Cyl } A & & \\ \uparrow x \mapsto b \otimes x & & \\ A & \xrightarrow{\varepsilon} & k \end{array} \tag{20}$$

As a vector space Con $A \cong A \oplus A[-1] \oplus k$. Denote by $T : A \to A[-1]$ the canonical map. Denote by e the element $1 \in k \subset$ Con A. For $u \in A$ we have:

$$\begin{aligned} \Delta_{\text{Con } A} u &= \Delta_A u \\ \Delta_{\text{Con } A} Tu &= (T \otimes \text{Id})\Delta_A u + e \otimes Tu; \\ \Delta_{\text{Con } A} e &= e \otimes e; \\ d_{\text{Con } A} u &= du; \\ d_{\text{Con } A} Tu &= u - Tdu - \varepsilon(u)e; \\ d_{\text{Con } A} e &= 0. \end{aligned}$$

For a morphism of coalgebras with counit $\phi : A \to$ Con B define a map $C\phi$: Con $A \to$ Con B as follows. Let $\phi : A \to B \oplus B[-1] \oplus k \cong$ Con B be defined by its components $f : A \to B$; $Tg : A \to B \to B[-1]$; $h : A \to k$. Then $C\phi : A \oplus A[-1] \oplus k \to B \oplus B[-1] \oplus k$ is defined by the matrix

$$\begin{pmatrix} f & 0 & 0 \\ Tg & \tilde{f} & 0 \\ h & 0 & \text{Id} \end{pmatrix} \tag{21}$$

, where $\tilde{f} : A[-1] \to B[-1]$ is the induced map. Note that if A and B are the chain complexes of polyhedral complexes, then the map $C\phi$ corresponds to the piecewise-linear continuation of ϕ such that the vertex of Con A goes to the vertex of Con B

PROPOSITION 6.9. *$C\phi$ is a homomorphism of dg coalgebras with counit.*

Proof. Preserving of counit is clear. To prove that $C\phi$ is a homomorphism, let us write explicitly the condition that ϕ is a homomorphism. We have

$$\begin{aligned} \Delta_{\text{Con } B}\phi(u) &= \Delta_{\text{Con } B}(f(u) + Tg(u) + h(u)) \\ &= \Delta_B f(u) + (T \otimes \text{Id})\Delta_B g(u) + e \otimes Tg(u) \\ &+ h(u) e \otimes e. \end{aligned}$$

$$\phi \otimes \phi(\Delta_A u) = (f + T \circ g + h) \otimes (f + T \circ g + h)\Delta_A u$$

The equality $\Delta_{\text{Con } B}\phi(u) = \phi \otimes \phi(\Delta_A u)$ yields:

$$f \otimes f \Delta_A u = \Delta_B f(u); g \otimes f \Delta_A u = \Delta_B g(u);$$

$$h \otimes f\Delta_A u = 0; f \otimes g\Delta_A u = 0;$$
$$g \otimes g\Delta_A u = 0; h \otimes g\Delta_A u = e \otimes g(u);$$
$$f \otimes h\Delta_A u = 0; g \otimes h\Delta_A u = 0;$$
$$h \otimes h\Delta_A u = h(u)e \otimes e.$$

1)For $u \in A \subset \text{Con } A$ we have $C\phi \otimes C\phi(\Delta_{\text{Con } A} u) = \phi \otimes \phi(\Delta_A u) = \Delta_{\text{Con } B} C\phi(u)$

2) For $Tu \in A[-1] \subset \text{Con } A$

$$\begin{aligned} C\phi \otimes C\phi(\Delta_{\text{Con } A} Tu) &= (C\phi \otimes C\phi)(T \otimes \text{Id}\Delta_A u + e \otimes Tu) \\ &= (\tilde{f} \otimes (f + Tg + h))(T \otimes \text{Id })\Delta_A u + (1 \otimes \tilde{f})(e \otimes Tu) \\ &= (Tf \otimes (f + Tg + h))\Delta_A u + (1 \otimes Tf)(e \otimes u) \\ &= (T \otimes \text{Id})\Delta_B f(u) + (1 \otimes Tf)(e \otimes u). \end{aligned} \tag{22}$$

$$\Delta_{\text{Con } B} C\phi(Tu) = \Delta_{\text{Con } B}(Tfu) = (T \otimes 1)\Delta_B f(u) + e \otimes Tf(u).$$

3) For $e \in k \subset \text{Con } A$, $\Delta_{\text{Con } B} C\phi(e) = \Delta_{\text{Con } B} e = e \otimes e$;
$C\phi \otimes C\phi\Delta_{\text{Con } A} e = C\phi \otimes C\phi e \otimes e = e \otimes e$.

4) The preservation of the differential and the counit can be checked directly. □

6.3.6. *Category of trees.* Let $\mathcal{T}_n^1$ be the category of planar trees with n inputs and at least one internal vertice. The morphisms are the contractions of internal edges. Let $F : \mathcal{T}_n^1 \to Vect$ be a functor such that for $t \in \mathcal{T}_n^1$,

$$F(t) = \bigotimes_{i \in vertices(t)} C_\bullet(K(v(i))), \tag{23}$$

where $v(i)$ is the multiplicity of i and for a contraction $c : t_1 \to t_2$, $F(c)$ is defined by the corresponding operadic maps. Here the tensor product 23 should be understood as a space of coinvariants

$$\Big(\bigoplus_\chi \bigotimes_i C_\bullet(K(v(\chi(i)))\Big)_{S_n},$$

where χ is any isomorphism $\{1, \dots, |vertices(t)|\} \to vertices(t)$. see [4]. Then by the construction of associahedra, $C_\bullet(\partial K_n) = \varinjlim_{\mathcal{T}_n^1} F$.

6.3.7. *Construction of coalgebra structure.* 1) $n = 0, 1, 2$. $C_\bullet(\mathcal{K}(n)) \cong k$ and we endow them with the standard coalgebra structure.

2) Suppose that we have found a structure of a dg coalgebra with counit on $C_\bullet(\mathcal{K}(i))$, $i < n$ such that all operadic maps $\circ_k^{i,j} : C_\bullet(\mathcal{K}(i)) \otimes C_\bullet(\mathcal{K}(j)) \to C_\bullet(\mathcal{K}(i + j - 1))$, $0 \le i, j, i + j - 1 \le n - 1$ are dg coalgebra morphisms and the counit maps $C_\bullet(\mathcal{K}(i)) \to k$ are the augmentation maps $\varepsilon|_{C_i(\mathcal{K}(m))} = 0$; $e(p) = 1$ for any 0-dimensional vertex of the decomposition. The functor F can be considered as

a functor $\mathcal{T}^1_n \to dgcoalg_1$. According to the section 6.3.4, $\varinjlim_{\mathcal{T}^1_n} F$ is canonically a coalgebra with counit and we define a coalgebra structure on $C_\bullet(\mathcal{K}(n))$ as the one on Con $\varinjlim_{\mathcal{T}^1_n} F \cong C_\bullet(\mathcal{K}(n))$. Let us check the induction assumption. All the maps $\circ^{i,j}_k$, $1 \le i,j \le n$, $i+j = n+1$ are automatically homomorphisms. The maps $\circ^{n,o}_j$ are produced from their restrictions $\mathrm{res}_{\circ^{n,0}_j} : C_\bullet(\partial\mathcal{K}(n)) \to C_\bullet(\mathcal{K}(n-1)) \cong$ Con $(C_\bullet(\partial\mathcal{K}(n-1)))$ in the following way: $\circ^{n,o}_j = C\mathrm{res}_{\circ^{n,0}_j}$. Therefore, since $\mathrm{res}_{\circ^{n,0}_j}$ are coalgebra morphisms by the induction assumption, so are $\circ^{n,o}_j$. We only need to check that the counit map is the augmentation map, which is obvious.

6.3.8. We define an operad of dg coalgebras with counit $\mathcal{A}$ by setting $\mathcal{A}(n)^i = C_{-i}(\mathcal{K}(n))$.Obviously, the differential in this operad has grading 1. Denote by $\mathcal{V}(n)$ the subspace in $\mathcal{A}(n)$ spanned by the images of the polyhedra of the decomposition of $\mathcal{K}(n)$ which do not lie on the boundary of $\mathcal{K}(n)$.

PROPOSITION 6.10. 1 *$\mathcal{A}$ is free as an operad of graded vector spaces and is freely generated by the subspaces $\mathcal{V}(n)$;*

2 *the counit map $\varepsilon : \mathcal{A} \to As$ is a quasiisomorphism of operads of dg coalgebras with counit.*

6.4. **An operad $\mathcal{F}$ and its cohomology.** We are going to define $\mathcal{F}$, a map $\mathcal{F} \to \mathcal{B}_\infty$, and to prove that the through map $\mathcal{F} \to \mathcal{B}_\infty \to e_2$ is a quasiisomorphism. We will construct the map $s : \mathcal{H}olie\{1\} \to \mathcal{F}$ in section 6.6 which will complete the proof of theorem 4.2.

6.4.1. We define $\mathcal{F} = \mathcal{O}(\mathcal{A})\{1\}$ and $\mathcal{G} = \mathcal{O}(\mathcal{A})$. The following proposition is an easy corollary of Propositions 6.8 and 6.10.

PROPOSITION 6.11. *An operad $\mathcal{G}$ is freely generated by operations $\phi(v)^1_{k_1\ldots k_n}$, $k_1,\ldots,k_1 \ge 1$ and D^1_k, $k > 1$. The map $\varepsilon : \mathcal{A} \to As$ produces maps $t' : \mathcal{G} \to \mathcal{B}$ and $t : \mathcal{F} \to \mathcal{B}_\infty$.*

6.4.2. It is clear that $H^\bullet(\mathcal{F}) = H^\bullet(\mathcal{G})\{1\}$. Therefore it suffices to compute $H^\bullet(\mathcal{G})$. We have through maps $r : \mathcal{F} \to \mathcal{B}_\infty \xrightarrow{k} e_2$ and $r' : \mathcal{G} \to \mathcal{B} \to e_2\{-1\}$.

PROPOSITION 6.12. *The maps r, r' are surjective on the level of cohomologies*

Proof. It suffices to prove these statements for the corresponding map $t'_* : H^\bullet(\mathcal{G}(2)) \to H^\bullet(e_2\{-1\})$. But $\mathcal{G}(2) \cong \mathcal{B}(2)$ and $k : \mathcal{B}(2) \to e_2\{-1\}(2)$ is a quasiisomorphism by Theorem 4.2 1. □

Therefore, to prove that r, r' are quasiisomorphisms, it suffices to give upper bounds for $H^\bullet(\mathcal{G})$. This can be achieved by means of spectral sequences.

6.4.3. *Filtrations on $\mathcal{G}$ and $P(\mathcal{A})$.* For the definition of $P(\mathcal{A})$ see Proposition 6.5.

Introduce the following filtration on $P(\mathcal{A})$: $F^0P(\mathcal{A})(n,m) = P(\mathcal{A})(n,m)$; $F^1(P(\mathcal{A})(n,m)) = 0$ if $n \geq m$ and $F^1(P(\mathcal{A})(n,m)) = P(\mathcal{A})(n,m)$ if $n > m$. $F^kP(\mathcal{A})(n,m)$ is defined as a span of compositions of $\geq k$ operations among which at least k belong to $F^1P(\mathcal{A})$. Note that since $P(\mathcal{A})$ is a PROP generated by an operad, $P(\mathcal{A})(n,m) = 0$, $n < m$ and $P(\mathcal{A})(n,n) = k[S_n]$.

The filtration F induces a filtration on $\mathcal{G}$ which is just a filtration by the number of internal vertices of trees presenting elements of $\mathcal{G}$. It is easy to see that such a filtration is preserved by the differential on $P(\mathcal{A})$. Before consideration of the spectral sequence of $\mathcal{G}$ associated with F we will prove some Lemmas.

6.4.4. *$\phi_n(a)$ up to $F^2P(\mathcal{A})$.* Let V be a $P(\mathcal{A})$-algebra (or $\mathcal{G}$-algebra, which is the same); $x_1, \ldots, x_n \in T^{\geq 1}V$; $a \in \mathcal{A}(n)$. Let $\phi_n^k : \mathcal{A}(n) \otimes TV^{\otimes n} \to TV \to V^{\otimes k}$ be the corestriction. Then

$$\phi_n^k(a, x_1, \ldots, x_n) = (\phi_n^1)^{\otimes k}(\Delta_k a_1, \Delta_k x_1 \ldots, \Delta_k x_n).$$

Let $\Delta_k^{i_1 \ldots i_k} : TV \to TV^{\otimes k} \to T^{i_1}V \otimes \ldots \otimes T^{i_k}V$ be the corestriction. Then

$$\phi_n^k(a, x_1, \ldots, x_n) = \sum_{i_1^1 \ldots i_k^n} (\phi_n^1)^{\otimes k}(\Delta_k a_1, \Delta_k^{i_1^1 \ldots i_k^1} x_1 \ldots, \Delta_k^{i_1^n \ldots i_k^n} x_n). \tag{24}$$

A summand in (24) belongs to $F^sP(\mathcal{A})$ if among the numbers

$$\sum_{r=1}^n i_1^r, \ldots, \sum_{r=1}^n i_k^r \tag{25}$$

at least s are greater than 1. Note that if some of $\sum_{r=1}^n i_q^r$ is equal to 0, then the corresponding summand is equal to zero. Therefore, to compute (24) up to $F^2P(\mathcal{A})$ one has to take the sum only over such $i_1^1 \ldots i_k^n$ that the numbers (25) are all equal to 1 except, maybe, one of them, say $\sum_{r=1}^n i_s^r$. The corresponding summand in (24) is equal to

$$\sigma = \sum_{j_0 \ldots j_n} \bigotimes_{l=1}^k \phi_n(\Delta_k^{j_0 l} a, \Delta_k^{j_1 l} x_1, \ldots, \Delta_k^{j_n l} x_n),$$

where

$$\Delta_k a = \sum_j \bigotimes_{l=1}^k \Delta_k^{jl} a;$$

$$\Delta_k^{i_1^p \ldots i_k^p} x_p = \sum_j \bigotimes_{l=1}^k \Delta_k^{jl} x_p.$$

For $l \neq s$, $\Delta_k^{j,l} x_r \in T^0 V$ for all r except one, say r_l. One may assume that $\Delta_k^{jl} x_r = 1$ for $r \neq r_l$, $l \neq s$. We will have

$$\sigma = \sum \otimes_{l=1}^{s-1} \phi_n(\Delta_k^{j_0 l} a, 1, 1 \ldots, \Delta_k^{j_{r_l}} x_{r_l} \ldots 1)$$
$$\otimes \phi_n(\Delta_k^{j_0 s} a, \Delta_k^{j^1 s} x_1, \ldots, \Delta^{j_n s} x_n)$$
$$\otimes \bigotimes_{l=1}^{s-1} \phi_n(\Delta_k^{j_0 l} a, 1, 1 \ldots, \Delta_k^{j_{r_l}} x_{r_l} \ldots 1). \tag{26}$$

But $\phi_n(a, 1, 1, \ldots, \xi, 1, \ldots, 1) = \varepsilon(a)\xi$, where $\xi \in V$, $x \in \mathcal{A}(n)$ and $\varepsilon \otimes \varepsilon \otimes \ldots \otimes 1 \otimes \varepsilon \otimes \ldots \otimes \varepsilon(\Delta_n a) = a$. Therefore we have

$$\sigma = \sum \bigotimes_{l=1}^{s-1} \Delta_k^{j_{r_l} l} x_{r_l} \otimes \phi_n(a \otimes \Delta_k^{j_1 s} x_1 \otimes \ldots \otimes \Delta_k^{j_n s} x_n)$$
$$\otimes \bigotimes_{l=s+1}^{k} \Delta_k^{j_{r_l} l} x_{r_l} \tag{27}$$

Let $y_i \in T^{k_i} V$. Set $\phi'_n(a, y_1, \ldots, y_n) = \phi_n(a, y_1, \ldots, y_n)$ if $k_1 + \ldots + k_n > 1$ and $\phi'_n(a, y_1, \ldots, y_n) = 0$ otherwise. One sees that the sum of such sigmas is equal to

$$\sum_{j_1 \ldots j_n} \pm x_1^{j_1 1} \cup x_2^{j_2 1} \cup \ldots \cup x_n^{j_n 1}$$
$$\otimes \phi'_n(a, x_1^{j_1 2}, x_2^{j_2 2}, \ldots, x_n^{j_n 2})$$
$$\otimes x_1^{j_1 3} \cup x_2^{j_2 3} \cup \ldots \cup x_n^{j_n 3} + \varepsilon(a) x_1 \cup \ldots \cup x_n, \tag{28}$$

where $\Delta_3 x_i = \sum_{j_i} x_i^{j_i 1} \otimes x_i^{j_i 2} \otimes x_i^{j_i 3}$, and $\cup$ means shuffle. Thus we have proven

LEMMA 6.13. *$\phi_n(x)$ is equal to (28) up to operations in $F^2 P(\mathcal{A})$.*

6.4.5. *Additional filtration on $F^1\mathcal{G}/F^2\mathcal{G}$.* Note that $\mathcal{W} = F^1\mathcal{G}/F^2\mathcal{G}$ is generated by the images of the elements $\phi(v)^1_{n_1 \ldots n_k}$, $v \in \mathcal{V}(k)$ and D^1_k. Define a filtration F' on $\mathcal{W}$ as follows. $F'_m \mathcal{W}$ is generated by $\phi(v)_{n_1 \ldots n_l}$, $l \leq m$ and D^1_l, $m \geq 1$.

LEMMA 6.14. 1) *Filtration F' is compatible with the induced differential on $\mathcal{W}$.*
2) *$d\phi(v)^1_{n_1 \ldots n_l} = \phi(\delta v)^1_{n_1 \ldots n_l}$ in $\mathrm{Gr}_{F'} \mathcal{W}$, where δ is the differential on $\mathcal{V}(l) \cong C_{-\bullet}(\mathcal{K}(k), \partial \mathcal{K}(k))$.*

Proof. Take $v \in V(k)$ corresponding to an element of decomposition of $\mathcal{K}(k)$ denoted by the same letter. Since v does not belong to $\partial \mathcal{K}(k)$, $v = \mathrm{Con}\ w$, $w \subset \partial \mathcal{K}(k)$, $w = \circ_i^{p,k-p}(\lambda, \mu)$, $\lambda, \mu \in \mathcal{K}(< k)$. We have $dv = \bar{\delta} v + w$, where d is the differential on

$\mathcal{A}$ and $\bar{\delta}v$ is the oriented sum of all faces of v that do not lie on $\partial\mathcal{K}(k)$. According to (16), for any $P(\mathcal{A})$-algebra, we have

$$\begin{aligned} &d\phi^1_{k_1\dots k_n}(v,x_1,\dots,x_n) - \phi^1_{k_1\dots k_n}(dv,x_1,\dots,x_n) \\ &\qquad - \sum(-1)^{|v|+|x_1|+\dots+|x_{l-1}|}\phi^1_{k_1\dots k_n}(v,x_1,\dots dx_l,\dots,x_n) \\ &= -\sum_s D^r_s\phi^s_{k_1\dots k_n}(v,x_1,\dots,x_n)+\sum(-1)^{|v|+|x_1|+\dots+|x_{l-1}|}\phi^1_{k_1\dots s\dots k_n}(v,x_1,\dots D^s_l x_l,\dots,x_n) \end{aligned} \tag{29}$$

Hence, up to $F^2P(\mathcal{A})$, we have

$$\begin{aligned} d\phi^1_{k_1\dots k_n}(v,x_1,\dots,x_n)&-\sum(-1)^{|v|+|x_1|+\dots+|x_{l-1}|}\phi^1_{k_1\dots k_n}(v,x_1,\dots,dx_l,\dots,x_n) \\ &= \phi^1_{k_1\dots k_n}(dv,x_1,\dots,x_n) = \phi^1_{k_1\dots k_n}(\bar{\delta}v,x_1,\dots,x_n) \\ &\qquad + \phi^1(\lambda,x_1,\dots x_{i-1},\phi(\mu,x_i,\dots,x_j),\dots,x_n) \end{aligned} \tag{30}$$

The last expression is equal to

$$\phi^1_{k_1\dots k_n}(\bar{\delta}v,x_1,\dots,x_n)$$

modulo $F'_{n-1}\mathcal{W}$ □

6.4.6. *Fundamental classes μ_k.* Let $\mu_k \in C_{k-2}(\mathcal{K}(k))$ be the fundamental class. It is clear that $\mu_k \in \mathcal{V}(k)$ and that

$$d\mu_k = \sum_{i=2}^{k-1}\mu_i\{\mu_{k+1-i}\}, \tag{31}$$

where

$$\mu_i\{\mu_j\} = \sum_{k=1}^{i}(-1)^{(k-1)(j-1)} \circ^{i,j}_k (\mu_i,\mu_j)$$

For a map $\chi : TV^{\otimes n} \to V$ define $T\chi : TV^{\otimes n} \to TV$ by the formula (cf. (28)

$$\begin{aligned} T\chi(x_1,\dots,x_n) = \sum_{j_1\dots j_n} &\pm x_1^{j_1 1}\cup x_2^{j_2 1}\cup\dots\cup x_n^{j_n 1} \\ &\otimes \chi(x_1^{j_1 2},x_2^{j_2 2},\dots,x_n^{j_n 2}) \\ &\otimes x_1^{j_1 3}\cup x_2^{j_2 3}\cup\dots\cup x_n^{j_n 3}. \end{aligned} \tag{32}$$

For $k>2$ denote by ϕ^1_k the corestriction of $\phi(\mu_k)$ on V. For $k=2$ let $\phi^1(\mu_2)$ be the restriction-corestriction of $\phi_2(\mu_2) : T^{\geq 1}V\otimes T^{\geq 1}V \to TV \to V$. Then we have

$$\phi_2(\mu_2)(x,y) = x\cup y + T\phi^1(\mu_2)(x,y) + k(x,y), \tag{33}$$

where $k \in F^2P(\mathcal{A})$. Finally, let ϕ_1 be the restriction-corestriction of the differential $D : T^{\geq 2}V \to TV \to V$.

LEMMA 6.15.

$$d\phi_k(x_1, \dots, x_k) = \sum_{i=1}^{k} \phi_i\{T\phi_{k+1-i}\} + \sum_r (-1)^{r-1}\phi_{k-1}(x_1, \dots x_r \cup x_{r+1}, \dots) + u, \tag{34}$$

where $u \in F^3\mathcal{G}$ and

$$\phi_i\{T\phi_{k+1-i}\}(x_1, \dots, x_k)$$
$$= \sum_{l=1}^{i} (-1)^{(l-1)(k-i)}(-1)^{(k+1-i)(|x_1|+\dots+|x_{l-1}|)}\phi_i(x_1, \dots, x_{l-1}, T\phi_{k+1-i}(x_l, \dots, x_{l+k-i}), \dots, x_k)$$

Proof. This immediately follows from (28), (31), and (33) □

6.5. **Spectral Sequence Associated with F.**

6.5.1. $E_1^{\bullet,\bullet}$. Since $\mathcal{G}$ is free, the Künneth formula implies that the operad of $E_1^{\bullet,\bullet}$ is a semi free operad generated by $H^\bullet(\mathcal{W})$. Let $\mathcal{Z} \subset \mathcal{W}$ be the S-submodule generated by the images of $\phi(\mu_k)^1_{n_1 \dots n_k}$.

LEMMA 6.16. 1) *$\mathcal{Z}$ is a subcomplex of $\mathcal{W}$.*
2) *$\mathrm{Gr}_{F'}\,\mathcal{Z} \to \mathrm{Gr}_{F'}\,\mathcal{W}$ is a quasiisomorphism.*
3) *$\mathcal{Z} \to \mathcal{W}$ is a quasiisomorphism*

Proof. 1). Follows from Lemma 6.15 which implies that

$$d\phi_k(x_1, \dots, x_k) = \sum_r (-1)^{r-1}\phi_{k-1}(x_1 \otimes \dots x_r \cup x_{r+1} \otimes \dots) + F^2\mathcal{G}. \tag{35}$$

2) Follows from Lemma 6.14.
3) Follows from 2). □

Consider the complex $\mathcal{Z}$. It is clear that the Schur functor $S_{\mathcal{Z}}V \cong \oplus_{n=2}^{\infty}\mathcal{Z}(n) \otimes_{S_n} T^nV \cong \bigoplus_{k=1}^{\infty} T^k(T^{\geq 1}V)[k-2] \cong T(T^{\geq 1}V[1])[-2]$ for any vector space V. The differential 35 coincides with the Hochschild differential of the algebra $(T^{\geq 1}V, \cup)$. Therefore,

$$S_{H^\bullet(\mathcal{Z})}V \cong HH_{2-\bullet}(T^{\geq 1}V) \cong$$
$$S^\bullet(T^{\geq 1}V/shuffles[1])[-2] \cong P(V)[-1] \cong S_{(e_2\{-1\})^*[-1]}V.$$

An S-module $\mathfrak{g} = (e_2\{-1\})^*[-1]$ constitutes the space of generators of the canonical resolution $\mathcal{HE}_2\{-1\}$

Therefore, $E_1^{\bullet,\bullet}$ is a free operad generated by $\mathfrak{g}$. We have a quasiisomorphism

$$\alpha : \mathcal{Z} \to H^\bullet(\mathcal{Z}) \tag{36}$$

such that the map $\alpha_*: S_{\mathcal{Z}}V \to S_{H^\bullet(\mathcal{Z})}V$ is the obvious projection $T(T^{\geq 1}V[1])[-2] \to S^\bullet(T^{\geq 1}V/shuffles[1])[-2]$

We are going to prove that the differential d_1 coincides with the differential on $\mathcal{HE}_2\{-1\}$. This implies that $E_2^{\bullet,\bullet} \cong e_2\{-1\}$ and that the map $r': \mathcal{G} \to e_2\{-1\}$ is a quasiisomorphism.

6.5.2. d_1. This differential is a quadratic differential on a free operad $F(\mathfrak{g})$ generated by $\mathfrak{g}$. Let $\mathfrak{g}(n) = e_2\{-1\}(n)^*[-1]$. The space of quadratic elements of $F(\mathfrak{g})$ is isomorphic to $\bigoplus \mathfrak{g}(i) \otimes \mathfrak{g}(j) \otimes_{S_j \times S_{i-1}} k[S_{i+j-1}]$, where we assume that the second factor is inserted into the first position of the first factor. Any such a differential defines on $\mathfrak{g}[1] = e_2\{-1\}^*$ a structure of cooperad. Let us denote the cooperad corresponding to d_1 by e'. We need to prove that $e' = e_2\{-1\}^*$.

Note that $e_2\{-1\}^*$ is cogenerated by $e_2\{-1\}^*(2)$. Therefore, it suffices to prove that the coinsertion maps $o_1^{2,n*}: e'(n+1) \to e'(2) \otimes e'(n)$ are the same as in $e_2\{-1\}^*$. For this we need to compute the restriction $\bar{d}_1: \mathfrak{g}(j+1) \to \mathfrak{g}(2) \otimes \mathfrak{g}(j) \otimes_{S_j} k[S_j]$. Take an operation $\xi = \varphi(\mu_l)^1_{k_1\ldots k_n}$. Then (31) allows one to compute $\bar{d}_1\alpha(\xi)$, where α is the map (36). After removing irrelevant terms, we obtain

$$\bar{d}_1\alpha(\xi(x_1,\ldots,x_l)) = \alpha\Big(D_2(T\phi_l(x_1,\ldots,x_l)) + \phi(\mu_2)_{1,1}(x_1, \phi_{l-1}(x_2\ldots x_l)) \pm \phi(\mu_2)_{1,1}(\phi_{l-1}(x_1\ldots x_{l-1}), x_l)\Big).$$

Let $\Delta x_i = \sum_{pq} \Delta^{p,q} x_i$, where $\Delta^{p,q}x_i \in T^pV \otimes T^qV$. Let $\Delta^{p,q}x_i = \sum_t x_{i1}^{pqt} \otimes x_{i2}^{pqt}$. Then

$$\begin{aligned}\bar{d}_1\alpha(\xi(x_1,\ldots,x_l)) = \alpha\Big(&\sum D_2(x_{r1}^{1,q} \otimes \phi^1(x_1,\ldots,x_{r2}^{1,q},\ldots,x_l)) \\ &+ \sum D_2(\phi^1(x_1,\ldots,x_{r1}^{q,1},\ldots,x_l) \otimes x_{r2}^{q,1}) \\ &+ \phi(\mu_2)_{1,1}(x_1,\phi_{l-1}(x_2\ldots x_l)) \pm \phi(\mu_2)_{1,1}(\phi_{l-1}(x_1\ldots x_{l-1}), x_l)\Big).\end{aligned}$$

Note that $\alpha(D_2(x\otimes y))$ is the cocommutator in $e_2\{-1\}^*$ and $\alpha(\phi(\mu_2)_{1,1}(x,y))$ is the coproduct. Now the coincidence of the cooperadic structures on e' and $e_2\{-1\}^*$ is obvious.

6.6. **Maps $s: \mathcal{H}olie\{1\} \to \mathcal{F}$ and $s': \mathcal{H}olie \to \mathcal{G}$.** We construct a map s satisfying the Theorem 4.2. We have a map of dg operads $\mathcal{H}oassoc \to \mathcal{A}$ such that $m_k \in \mathcal{H}oassoc \mapsto \mu_k \in \mathcal{A}$, where μ_k are the fundamental classes, see (6.4.6). Also we have a map $\mathcal{H}olie \to \mathcal{H}oassoc: m_k(x_1\ldots x_k) \to \sum_\sigma (-1)^\sigma \mu_n(x_{\sigma_1}\ldots x_{\sigma_n})$. Therefore, any $\mathcal{O}(\mathcal{A}) = \mathcal{G}$-structure on V implies a structure of a $\mathcal{H}olie$- algebra on TV. The operations are given by the antisymmetrization of $\phi(\mu_k)(x_1\ldots x_k)$.

LEMMA 6.17. *The operations $\phi^r(\mu_k)(x_1\ldots x_k)$, $k \geq 3$ and $\phi^r(\mu_2)(x_1 \otimes x_2) \mp \phi^r(\mu_2)(x_2\otimes x_1)$ are equal to zero whenever all x_i are in T^1V and $r \geq 2$.*

Proof. 1) $k \geq 3$. We have

$$\phi^r(\mu_k)(x_1 \dots x_k)$$
$$= (\phi^1)^{\otimes r}(\Delta_r\mu_k, \Delta_r x_1, \dots, \Delta_r x_k)$$
$$= \sum \bigotimes_{i=1}^{r} \phi^1(\Delta_r^{j_0 i}\mu_k, \Delta_r^{j_1 i}x_1, \dots, \Delta_r^{j_k i}x_k). \quad (37)$$

This is a sum of products. Take one of these products. Let the i-th multiple in it have $k - s_i$ arguments to be equal to 1 and s_i arguments from V. We have $\sum s_i = k$. This product can be rewritten as a product of operations $\phi(a_i)$, where $a_i \in \mathcal{K}(s_i)$. Since dim $\mathcal{K}(i) = i - 2$ if $i \geq 2$ and $\dim \mathcal{K}(1) = 0$, the total grading of such an operation is at least $-(s_1 - 1 + \dots + s_r - 1) = -(k - r)$. Since the degree of μ_k is $-(k-2)$, the product vanishes if $r \geq 3$. If $r = 2$, then the equality of the degrees is only possible when $s_1 = s_2 = 1$, that is when $k = 2$.

2)$k = 2$. The above argument allows us to assume $r = 2$. We have $\phi^2(\mu_2 \otimes x \otimes y) \mp \phi^2(\mu_2 \otimes x \otimes y) = \phi^1 \otimes \phi^1((\mu_2 \otimes \mu_2), ([x, y]), (1 \otimes 1)) - \phi^1 \otimes \phi^1((\mu_2 \otimes \mu_2), (1 \otimes 1), ([x, y])) = 0$. $\square$

Hence, the $\mathcal{H}olie$-algebra on TV can be restricted to V. Therefore, we have a map $s' : \mathcal{H}olie \to \mathcal{G}$. To compute the through map $\mathcal{H}olie\{1\} \to \mathcal{F} \to \mathcal{B}_\infty$, let us notice that the image of μ_k under the counit map $\mathcal{A} \to As$ is zero for $k > 2$. Therefore all higher products will go to zero, and $\phi^1(\mu_2, x, y) \mp \phi^1(\mu_2, y, x)$ will go to $m_{1,1}(x, y) \mp m_{1,1}(y, x)$, which proves that the diagram in Theorem 4.2 commutes.

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