Let $\alpha, \beta$ be the two complex roots of the equation
\[ x^2 - (1 + i)x + 1 = 0. \]

1) Show that $\alpha\beta$ lies on the unit circle, but is not a root of unity.

2) Show that the minimal polynomial of $\alpha$ and $\beta$ over $\mathbb{Z}$ is
\[ x^4 - 2x^3 + 4x^2 - 2x + 1. \]

3) Show that the vectors $(1, 1), (\alpha, \beta), (\alpha^2, \beta^2), (\alpha^3, \beta^3)$ generate a lattice $\Lambda \subset \mathbb{C}^2$.

Let now $X = \mathbb{C}^2/\Lambda$, a complex torus, and let $\Omega$ be the nowhere vanishing holomorphic 2-form on $X$ induced by $dz_1 \wedge dz_2$ on $\mathbb{C}^2$.

4) Show that the automorphism of $\mathbb{C}^2$ given by multiplication by
\[
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix}
\]
descends to an automorphism $f$ of $X$, which satisfies $f^*\Omega = \alpha\beta\Omega$.

Define now a holomorphic $\mathbb{Z}^2$-action on $X \times \mathbb{C}$ by
\[
(1, 0) \cdot (x, z) = (x, z + 1), \quad (0, 1) \cdot (x, z) = (f(x), z + i),
\]
where $(x, z) \in X \times \mathbb{C}$.

5) Show that the quotient $Y = (X \times \mathbb{C})/\mathbb{Z}^2$ is a compact complex manifold, and that the map $(x, z) \mapsto z$ induces a holomorphic submersion $\pi : Y \to Z$, where $Z = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ is a complex torus, with all fibers of $\pi$ biholomorphic.
6) Show that $$H^0(Y, K_Y^\ell) = 0,$$
for all $$\ell \geq 1.$$ 

(Hint: given a nontrivial section, pull it back to $$X \times \mathbb{C}$$ and compare it with 
$$(\Omega \wedge dz)^\otimes\ell$$)

7) Show that the volume form
$$i\Omega \wedge \overline{\Omega} \wedge dz \wedge d\overline{z}$$
on $$X \times \mathbb{C}$$ is invariant under the $$\mathbb{Z}^2$$-action, and so descends to a volume form $$V$$ on $$Y.$$  

8) Show that $$V$$ defines a Hermitian metric $$h$$ on $$K_Y^*$$ with curvature $$R_h$$ identically zero. Deduce that $$c_1(Y) = 0$$ in $$H^2(Y, \mathbb{R}).$$

Lastly, we show that $$Y$$ is not Kähler. Assume for a contradiction that $$Y$$ admits a Kähler metric $$\omega_Y,$$ and consider the class
$$[\alpha] := [\omega_Y|_X] \in H^{1,1}(X, \mathbb{R}),$$
where $$X \subset Y$$ is any chosen fiber of $$\pi.$$

9) Show that $$f^* [\alpha] = [\alpha].$$

For the last question, you can use without proof the (easy) fact that the class $$[\alpha]$$ on the torus $$X$$ contains a unique “Euclidean” Kähler metric $$\omega_X,$$ i.e. which is induced by a Euclidean metric on $$\mathbb{C}^2.$$ You can also use Bieberbach’s theorem.

10) Show that $$f^* \omega_X = \omega_X,$$
i.e. $$f$$ acts as an isometry of $$(X, \omega_X).$$ Show that this implies that $$f$$ has finite order, and that this is impossible.

2