1) Let $E \to M$ be a vector bundle. Show that the bundle $E^* \otimes E$ has a global smooth section which is never vanishing.

**Solution.** We have that $E^* \otimes E \cong \text{Hom}(E, E) = \text{End}(E)$, the bundle of endomorphisms of $E$. The identity $\text{Id} : E \to E$ is such an endomorphism, and viewed as a section of $E^* \otimes E$ it is never vanishing.

2) Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Define an operation on $\mathfrak{g} \times G$ by

$$(X, g) \cdot (Y, h) = (X + d(C_g)e(Y), gh),$$

where $C_g(h) = ghg^{-1}$ is conjugation by $g$.

(a) Show that this makes $\mathfrak{g} \times G$ into a Lie group.

Let $\mu : \mathfrak{g} \times G \to TG$ be the map defined by

$$\mu(X, g) = (g, d(R_g)e(X)).$$

(b) Show that $\mu$ a diffeomorphism.

**Solution.** (a) Clearly the operation is a smooth map. The element $(0, e)$ is clearly the identity. The inverse of $(X, g)$ is

$$(-d(C_{g^{-1}})e(X), g^{-1}),$$

since

$$(X - d(C_g)e)(d(C_{g^{-1}})e(X), gg^{-1}) = (0, e) = (-d(C_{g^{-1}})e(X) + d(C_{g^{-1}})e(X), g^{-1}g).$$

The inverse is also clearly smooth. Lastly, for any $(X, g), (Y, h), (Z, k)$ we have

$$(X, g) \cdot ((Y, h) \cdot (Z, k)) = (X, g) \cdot (Y + d(C_{h})e(Z), hk) = (X + d(C_{g})e(Y) + d(C_{gh})e(Z), ghk),$$

$$((X, g) \cdot (Y, h)) \cdot (Z, k) = (X + d(C_{g})e(Y), gh) \cdot (Z, k) = (X + d(C_{g})e(Y) + d(C_{gh})e(Z), ghk),$$

a required.

(b) $\mu$ is clearly smooth. It is bijective since its inverse is given by

$$\mu^{-1}((g, X_g)) = (d(R_{g^{-1}})g(X_g), g),$$
which is also smooth.

3) Let $\omega = dx + xdy \in \Lambda^1 \mathbb{R}^3$, and let $D = \ker \omega \subset T\mathbb{R}^3$ be the smooth distribution in $\mathbb{R}^3$ given by

$$D_p = \{X_p \in T_p \mathbb{R}^3 \mid \omega_p(X_p) = 0\},$$

for all $p \in \mathbb{R}^3$. Show that $D$ is integrable.

**Solution.** From the practice problem 3, we know that $D$ is integrable iff there exists a 1-form $\alpha \in \Lambda^1 \mathbb{R}^3$ with

$$d\omega = \alpha \wedge \omega.$$

We have

$$d\omega = d(dx + xdy) = dx \wedge dy,$$

so $\alpha = -dy$ works, since

$$\alpha \wedge \omega = -dy \wedge (dx + xdy) = -dy \wedge dx = d\omega.$$

4) Let $p, q$ be two positive integers and consider the $\mathbb{C}$-action on $(\mathbb{C}^p \setminus \{0\}) \times (\mathbb{C}^q \setminus \{0\})$ given by

$$t \cdot (z_1, \ldots, z_p, w_1, \ldots, w_q) = (e^t z_1, \ldots, e^t z_p, e^{it} w_1, \ldots, e^{it} w_q),$$

where $t \in \mathbb{C}$, and $z = (z_1, \ldots, z_p) \in \mathbb{C}^p \setminus \{0\}$, $w = (w_1, \ldots, w_q) \in \mathbb{C}^q \setminus \{0\}$.

(a) Show that this action is free (i.e. that if $t \cdot (z, w) = (z, w)$ for some $t \in \mathbb{C}$ and some $(z, w) \in (\mathbb{C}^p \setminus \{0\}) \times (\mathbb{C}^q \setminus \{0\})$, then $t = 0$).

Call then $X_{p,q}$ the quotient smooth manifold (you do not need to prove that it is a manifold).

(b) Show that $X_{p,q}$ is diffeomorphic to $S^{2p-1} \times S^{2q-1}$, a product of odd-dimensional spheres.

**Solution.** (a) Let us use the more compact notation

$$t \cdot (z, w) = (e^t z, e^{it} w).$$

Then if this equals $(z, w)$ we must have $e^t = 1$ so $t = 2\pi ik$ for some $k \in \mathbb{Z}$, but also $e^{it} = 1$ so $t = 2\pi \ell$ for some $\ell \in \mathbb{Z}$, which implies that $t = 0$, so the action is free.
(b) If we write $t = x + iy$ with $x, y \in \mathbb{R}$, then

$$t \cdot (z, w) = (e^x e^{iy} z, e^{-y} e^{ix} w),$$

and $|e^x e^{iy} z| = 1$ iff $x = -\log |z|$ while $|e^{-y} e^{ix} w| = 1$ iff $y = \log |w|$. Therefore with these choices of $x, y$ we see that $t \cdot (z, w)$ lies on the product of the unit spheres $S^{2p-1} \times S^{2q-1}$. The map we just defined

$$(z, w) \mapsto \left( e^{i \log |w|} \frac{z}{|z|}, e^{-i \log |z|} \frac{w}{|w|} \right) \in S^{2p-1} \times S^{2q-1},$$

clearly descends to a smooth and bijective map $X_{p,q} \rightarrow S^{2p-1} \times S^{2q-1}$ with smooth inverse induced by the inclusion

$$S^{2p-1} \times S^{2q-1} \subset (\mathbb{C}^p \setminus \{0\}) \times (\mathbb{C}^q \setminus \{0\}),$$

and therefore it gives a diffeomorphism.

5) Let $M$ be a smooth manifold and $\nabla$ be a connection on $TM$, with torsion

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Show that

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} T(X, Y),$$

defines a new connection on $TM$ with vanishing torsion.

**Solution.** The torsion $T(X, Y)$ is a tensor of type $(1, 2)$. This implies that it is $C^\infty(M)$-linear in both arguments $X$ and $Y$. It is also skew-symmetric, i.e. $T(X, Y) = -T(Y, X)$. Since $\nabla$ is a connection and $T(X, Y)$ is a tensor, it follows immediately that $\tilde{\nabla}$ is also a connection. Its torsion equals

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \nabla_X Y - \nabla_Y X - \frac{1}{2} T(X, Y) + \frac{1}{2} T(Y, X) - [X, Y] = 0,$$

as required.