445-2 Differential Geometry  
Northwestern University  
Solution of Homework 1

1) Let $p, q$ be two positive integers and consider the $\mathbb{C}$-action on $(\mathbb{C}^p \backslash \{0\}) \times (\mathbb{C}^q \backslash \{0\})$ given by

$$t \cdot (z_1, \ldots, z_p, w_1, \ldots, w_q) = (e^t z_1, \ldots, e^t z_p, e^{it} w_1, \ldots, e^{it} w_q),$$

where $t \in \mathbb{C}$, and $(z_1, \ldots, z_p) \in \mathbb{C}^p \backslash \{0\}, (w_1, \ldots, w_q) \in \mathbb{C}^q \backslash \{0\}$.

(a) Show that this action is free.

Call then $X_{p,q}$ the quotient complex manifold.

(b) Show that $X_{p,q}$ is diffeomorphic to $S^{2p-1} \times S^{2q-1}$, a product of odd-dimensional spheres.

**Solution.** (a) Let us use the more compact notation

$$t \cdot (z, w) = (e^t z, e^{it} w).$$

Then if this equals $(z, w)$ we must have $e^t = 1$ so $t = 2\pi ik$ for some $k \in \mathbb{Z}$, but also $e^{it} = 1$ so $t = 2\pi \ell$ for some $\ell \in \mathbb{Z}$, which implies that $t = 0$, so the action is free.

(b) If we write $t = x + iy$ with $x, y \in \mathbb{R}$, then

$$t \cdot (z, w) = (e^x e^{iy} z, e^{-y} e^{ix} w),$$

and $|e^x e^{iy} z| = 1$ iff $x = -\log |z|$ while $|e^{-y} e^{ix} w| = 1$ iff $y = \log |w|$. Therefore with these choices of $x, y$ we see that $t \cdot (z, w)$ lies on the product of the unit spheres $S^{2p-1} \times S^{2q-1}$. The map we just defined

$$(z, w) \mapsto \left(e^{i\log |w|} \frac{z}{|z|}, e^{-i\log |z|} \frac{w}{|w|}\right) \in S^{2p-1} \times S^{2q-1},$$

clearly descends to a smooth and bijective map $X_{p,q} \to S^{2p-1} \times S^{2q-1}$ with smooth inverse induced by the inclusion

$$S^{2p-1} \times S^{2q-1} \subset (\mathbb{C}^p \backslash \{0\}) \times (\mathbb{C}^q \backslash \{0\}),$$

and therefore it gives a diffeomorphism.
2) Let \( X \) be a compact complex manifold of complex dimension \( n \) and assume that \( f : \mathbb{CP}^n \to X \) is a holomorphic map which is also a finite sheeted covering. Prove that \( X \) must be biholomorphic to \( \mathbb{CP}^n \). Then find a compact real manifold \( Y \) of dimension \( 2n \) and a nontrivial finite sheeted smooth covering map \( f : \mathbb{CP}^n \to Y \) for any \( n \) odd.

**Solution.** Assume that \( f : \mathbb{CP}^n \to X \) is a finite sheeted holomorphic covering map. Then the deck transformations of \( f \) act on \( \mathbb{CP}^n \) as biholomorphisms. But we know that \( \text{Aut}(\mathbb{CP}^n) = \text{PGL}(n+1, \mathbb{C}) \), i.e. all biholomorphisms of \( \mathbb{CP}^n \) are induced by invertible linear transformations of \( \mathbb{C}^{n+1} \). Any such transformation is represented by a matrix \( A \in \text{GL}(n+1, \mathbb{C}) \), which therefore has a nonzero eigenvector \( v \), so \( Av = \lambda v \) for some nonzero complex number \( \lambda \). This means that the induced map on \( \mathbb{CP}^n \) fixes the point that corresponds to \( v \). But a deck transformation of a nontrivial covering space does not have fixed points unless it is the identity, and we are done.

The easiest counterexample when \( Y \) is not complex is \( Y = \mathbb{RP}^2 \) double covered by \( S^2 = \mathbb{CP}^1 \). Obviously the reflection on \( \mathbb{CP}^1 \) is not holomorphic since it reverses the orientation. Similarly we construct a “reflection” on \( \mathbb{CP}^{2n+1} \) by sending \([z_0 : \cdots : z_{2n+1}]\) to \([\overline{z}_1 : -\overline{z}_0 : \overline{z}_2 : \cdots : \overline{z}_{2n+1} : -\overline{z}_{2n}]\). This is a smooth map without fixed points, so the quotient is a smooth manifold \( Y \) double covered by \( \mathbb{CP}^{2n+1} \).

There are no counterexamples for \( n \) even, since any covering space \( f : \mathbb{CP}^{2n} \to X \) is trivial. This is because every continuous map from \( \mathbb{CP}^{2n} \) to itself has a fixed point (using the Lefschetz fixed point theorem).

3) Let \( X = \mathbb{C}^2/\Lambda \) be the complex torus given by the square lattice \( \Lambda \) generated by \((1,0), (i,0), (0,1), (0,i)\) over \( \mathbb{Z} \). Consider the holomorphic map \( \sigma : \mathbb{C}^2 \to \mathbb{C}^2 \) given by

\[
\sigma(z,w) = \left( z + \frac{1}{2}, -w \right).
\]

(a) Show that \( \sigma \) induces a holomorphic involution \( \sigma : X \to X \) that has no fixed points.

It follows that the quotient \( Y = X/\sigma \) is a compact complex manifold with complex dimension 2.

(b) Show that there is no nonzero holomorphic 2-form on \( Y \).

(c) Show that there is a never-vanishing smooth \((2,0)\)-form on \( Y \).
Solution. (a) Obviously $\sigma((z,w)+\Lambda) \subset \sigma(z,w)+\Lambda$, so we get a holomorphic map $\sigma:X \to X$. Clearly $\sigma \circ \sigma(z,w) = (z+1,w)$ which is the same as $(z,w)$ in the torus, so $\sigma$ is an involution. If we had $\sigma(z,w) \in (z,w)+\Lambda$, then $z + \frac{1}{2}$ would differ from $z$ by a Gaussian integer, which is false, so $\sigma$ has no fixed points.

(b) Call $\pi:X \to Y$ the quotient map. If $\varphi$ is a holomorphic 2-form on $Y$, then $\pi^*\varphi$ is a holomorphic 2-form on $X$. But on $X$ we have the never-vanishing holomorphic 2-form $dz \wedge dw$, so the quotient

$$\frac{\pi^*\varphi}{dz \wedge dw}$$

is a global holomorphic function on $X$, which is therefore constant. So $\pi^*\varphi = cdz \wedge dw$, and since $\pi^*\varphi$ is the pullback of a form from $Y$ we have $\sigma^*\pi^*\varphi = \pi^*\varphi$. This implies

$$cdz \wedge dw = c\sigma^*(dz \wedge dw) = -cdz \wedge dw,$$

which shows that $c = 0$ and $\pi^*\varphi = 0$. But $\pi$ is a holomorphic covering, so a local isomorphism, and therefore $\varphi = 0$.

(c) Write $z = x + iy, w = s + it$ with $x,y,s,t \in \mathbb{R}$. The never-vanishing smooth $(2,0)$-form $e^{2\pi ix}dz \wedge dw$ on $X$ (which is well-defined) is invariant under $\sigma$ because

$$\sigma^*(e^{2\pi ix}dz \wedge dw) = -e^{2\pi ix+\pi i}dz \wedge dw = e^{2\pi ix}dz \wedge dw,$$

and so it descends to $Y$. 